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Correlation Between Mesh Geometry and Stiffness Matrix Conditioning for Nonlocal Diffusion Models

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Abstract. Nonlocal diffusion models involve integral equations that account for nonlocal interactions and do not explicitly employ differential operators in the space variables. Due to the nonlocality, they might look different from classical partial differential equation (PDE) models, but their local limit reduces to partial differential equations. The effect of mesh element anisotropy, mesh refinement and kernel functions on the conditioning of the stiffness matrix for a nonlocal diffusion model on 2D geometric domains is considered, and the results compared with those obtained from typical local PDE models. Numerical experiments show that the condition number is bounded by $c\delta^{-2}$ (where c is a constant) for an integrable kernel function, and is not affected by the choice of the basis function. In contrast to local PDE models, mesh anisotropy and refinement affect the condition number very little.

AMS subject classifications: Peridynamics, condition number, finite element method.

Key words: 65R20, 65M60

1. Introductions

Nonlocal diffusion equations and nonlocal peridynamic models have received considerable attention in recent years. Peridynamic theory was developed by Silling [1]. Nonlocal diffusion and peridynamic theory involve integral equations rather than differential equations to model cracked surfaces and deformations, and have also been extensively applied elsewhere — e.g. to turbulence [2], porous flow [3], nanofibers [4, 5], and fracture and damage modelling of membranes [4]. Refs. [6, 7] provide recent surveys of nonlocal diffusion and peridynamic models, and their applications. It has been shown that a nonlocal peridynamic model reduces to a classical local model (such as in elasticity theory) when the length scale (horizon) goes to zero [8]. The effect of various kernel functions on the nonlocal advection problem has been investigated for a 1D problem [9]. Researchers have also studied finite difference and finite element discretisation of nonlocal diffusion and

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peridynamic models [10–13], including *a posteriori* error analysis and the connection between the horizon and the condition number in Ref. [10]. Condition number estimates and upper bounds for the discretised linear system, and the effect of the horizon and mesh size on the condition number for isotropic elements, have been investigated [14]. Interactions between mesh geometry, mesh refinement and the condition number of the global stiffness matrix for classical PDE have also been considered. Thus the connection between the anisotropy of mesh elements and the condition number for elliptic PDE was investigated in Ref. [15]; various mesh quality metrics, interpolation error and the condition number for elliptic, parabolic and hyperbolic PDE were explored in Ref. [16]; and connections between the mesh quality metric, preconditioner and the linear solver for elliptic PDE were studied computationally in Refs. [17, 18].

In this article, the Galerkin finite element method is used to discretise a linear nonlocal diffusion system, in order to study the effect of the anisotropy of the mesh element (element shape), mesh refinement (element size) and kernel functions on the condition number for a nonlocal diffusion model on 2D geometric domains. There are various nonlocal models, such as a bond-based model [1, 13] and a state-based model [19]. We consider a bond-based nonlocal model that involves central forces between particles [1, 13, 20], and numerically demonstrate the effect of an integrable kernel function on the condition number for both piecewise linear and piecewise constant basis functions. Conditioning is important, because it affects the accuracy of the solution and the convergence rate in solving the discretised linear system [15, 16]. This article is the first to explore the connections between anisotropy, mesh refinement and the condition number for 2D meshes with various kernel functions for a scalar nonlocal diffusion model. This is computationally challenging for various reasons. First, two different quadrature rules are needed to approximate the double integral terms on 2D geometric domains, to avoid the singularity of the denominator when the condition number of the global stiffness matrix is computed. Second, it is desirable to compute approximately the area of intersection between the horizon (δ) and the triangular element when the quadrature rule is used. Finally, the number of intersections between the horizon and the triangular element increases significantly as the level of mesh refinement increases, and so is computationally expensive.

The conditioning of the stiffness matrix is investigated for both piecewise constant and piecewise linear basis functions, assuming an integrable kernel function. In each case, the effect of changing the anisotropy and mesh size (h) on the conditioning of the scalar nonlocal diffusion model is examined. The analytical results show that the condition number is bounded by $c\delta^{-2}$ (where c is a constant) when a finite integrable kernel function is employed [14]. For an integrable kernel function, it is shown numerically in each case that the condition number is barely affected by the choice of basis function. The constant c in the condition number bound ($c\delta^{-2}$) is computed for uniform triangular and rectangular meshes in 2D. For general elliptic PDE, it is well-known that the condition number is proportional to h^{-2} when the mesh has the same anisotropy on uniform triangular and rectangular meshes in 2D [16]. Mesh anisotropy also affects the condition number for general elliptic PDE — e.g. if θ is the smallest angle in the right triangle, then the condition number is proportional to $\sin^{-1}(2\theta)$ [15] such that the condition number sharply increases