East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.090614.031014a

Vol. **4**, No. 4, pp. 329-344
November 2014

Backward Error Analysis for an Eigenproblem Involving Two Classes of Matrices

Lei Zhu[1] and Weiwei Xu[2,*]

[1] *College of Civil Aviation, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, P. R. China;*
College of Engineering, Nanjing Agricultural University, Nanjing 210031, P. R. China.
[2] *School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, P. R. China.*

Received 9 June 2014; Accepted (in revised version) 3 October 2014

Available online 5 November 2014

Abstract. We consider backward errors for an eigenproblem of a class of symmetric generalised centrosymmetric matrices and skew-symmetric generalised skew-centrosymmetric matrices, which are extensions of symmetric centrosymmetric and skew-symmetric skew-centrosymmetric matrices. Explicit formulae are presented for the computable backward errors for approximate eigenpairs of these two kinds of structured matrices. Numerical examples illustrate our results.

AMS subject classifications: 65F10, 65F15

Key words: Backward error, symmetric generalised centrosymmetric matrix, skew-symmetric generalised skew-centrosymmetric matrix.

1. Introduction

It is well-known that backward errors are very important for assessing the stability and quality of numerical algorithms. In this article, we consider backward errors for an eigenproblem of a special class of symmetric generalised centrosymmetric matrices and skew-symmetric generalised skew-centrosymmetric matrices, with practical applications. For example, a small perturbation method and backward errors for an eigenproblem were key techniques for a nonlinear component level model, and a state variables linear model of a turbofan engine — cf. [16–18].

Let $\mathscr{C}$ and $\mathscr{C}^{m\times n}$ denote the set of complex numbers and $m \times n$ complex matrices, respectively. (We will abbreviate $\mathscr{C}^{m\times 1}$ as $\mathscr{C}^{m}$.) The conjugate, transpose, conjugate transpose and Moore-Penrose generalised inverse of a matrix A are denoted by $\bar{A}$, A^T, A^* and A^+, respectively. The identity matrix of order n is denoted by I_n; the matrix norm adopted is

*Corresponding author. *Email addresses:* nuaazhulei@gmail.com (L. Zhu), wwx19840904@sina.com (W. Xu)

the Frobenius norm defined by $\|A\|_F = \sqrt{tr(A^*A)}$; and P_A and $P_A^{\perp}$ denote the orthogonal projection onto $\mathscr{R}(A)$ and the projection complementary to P_A, respectively. We also write $\mathscr{O}\mathscr{C}^{m\times m} = \{A \in \mathscr{C}^{m\times m} | A^T A = AA^T = I_m\}$.

Definition 1.1 (cf. Ref. [1]). Let $A, B \in \mathscr{C}^{k\times k}$, $\mu, \nu \in \mathscr{C}^k$, $\beta \in C$ and assume $P \in \mathscr{C}^{k\times k}$ is nonsingular. Then the block matrices

$$\mathscr{A}_{2k} = \begin{pmatrix} A & BP \\ P^{-1}B & P^{-1}AP \end{pmatrix} \qquad (k \geq 1),$$

$$\mathscr{A}_{2k+1} = \begin{pmatrix} A & \mu & BP \\ \nu^T & \beta & \nu^T P \\ P^{-1}B & P^{-1}\mu & P^{-1}AP \end{pmatrix} \ (k \geq 0),$$

are called $2k$, $2k+1$ step generalised centrosymmetric matrices and denoted by $\mathscr{G}\mathscr{C}^{2k\times 2k}$ and $\mathscr{G}\mathscr{C}^{(2k+1)\times(2k+1)}$, respectively. Similarly,

$$\mathscr{B}_{2k} = \begin{pmatrix} A & BP \\ -P^{-1}B & -P^{-1}AP \end{pmatrix} \qquad (k \geq 1),$$

$$\mathscr{B}_{2k+1} = \begin{pmatrix} A & \mu & BP \\ -\nu^T & \beta & \nu^T P \\ -P^{-1}B & -P^{-1}\mu & -P^{-1}AP \end{pmatrix} \ (k \geq 0),$$

are called $2k$, $2k+1$ step generalised skew-centrosymmetric matrices and denoted by $\mathscr{G}\tilde{\mathscr{C}}^{2k\times 2k}$ and $\mathscr{G}\tilde{\mathscr{C}}^{(2k+1)\times(2k+1)}$, respectively.

Definition 1.2 (cf. Ref. [6]). We define $\mathscr{S}\mathscr{G}\mathscr{C}^{m\times m} = \{A \in \mathscr{G}\mathscr{C}^{m\times m} | A = A^T\}$ and $\tilde{\mathscr{S}}\mathscr{G}\tilde{\mathscr{C}}^{m\times m} = \{A \in \mathscr{G}\tilde{\mathscr{C}}^{m\times m} | A = -A^T\}$ — i.e. as the sets of symmetric generalised centrosymmetric matrices and skew-symmetric generalised skew-centrosymmetric matrices, respectively.

In Definition 1.1, P is restricted to be orthogonal; and the corresponding classes of symmetric generalised centrosymmetric matrices and skew-symmetric generalised skew-centrosymmetric matrices are denoted by $\mathscr{K}_1$ and $\mathscr{K}_2$, respectively. These classes of symmetric generalised centrosymmetric matrices and skew-symmetric generalised skew-centrosymmetric matrices have practical applications in aerostatics, information theory, linear system theory, and linear estimate theory [1–6]. We can obtain the block forms of $\mathscr{K}_1$ and $\mathscr{K}_2$ as follows (for a proof see Lemmas 2.3 and 2.6 below):
for $2k$ ($k \geq 1$),

$$\mathscr{K}_1 = \left\{ \begin{pmatrix} A_1 & BP_0 \\ P_0^{-1}B & P_0^{-1}A_1P_0 \end{pmatrix} \right\}, \quad \mathscr{K}_2 = \left\{ \begin{pmatrix} A_2 & BP_0 \\ -P_0^{-1}B & -P_0^{-1}A_2P_0 \end{pmatrix} \right\};$$

for $2k+1$ ($k \geq 0$),

$$\mathscr{K}_1 = \left\{ \begin{pmatrix} A_1 & \mu & BP_0 \\ \mu^T & \beta & \mu^T P_0 \\ P_0^{-1}B & P_0^{-1}\mu & P_0^{-1}A_1P_0 \end{pmatrix} \right\},$$

$$\mathscr{K}_2 = \left\{ \begin{pmatrix} A_2 & \mu & BP_0 \\ -\mu^T & 0 & \mu^T P_0 \\ -P_0^{-1}B & -P_0^{-1}\mu & -P_0^{-1}A_2 P_0 \end{pmatrix} \right\} ,$$

where $A_1, A_2, B, P_0 \in \mathscr{C}^{k\times k}$, $\mu \in \mathscr{C}^k$ satisfy $A_1 = A_1^T, A_2 = -A_2^T, B = B^T, P_0^T P_0 = I_k$.

Definition 1.3 (cf. Ref. [6]). If e_i denotes the i-th column of the identity matrix, then $P_0 = (e_m, e_{m-1}, \cdots, e_1)$ is called an m-step sub-identity matrix.

It is not difficult to see that $\mathscr{K}_1$ and $\mathscr{K}_2$ depend upon the orthogonal matrix P_0. If P_0 is a sub-identity matrix, then $\mathscr{K}_1$ and $\mathscr{K}_2$ reduce to the sets of well-known symmetric centrosymmetric matrices and skew-symmetric skew-centrosymmetric matrices, respectively. Throughout this article, we always assume that orthogonal matrix P_0 is fixed.

It is well known that structured eigenvalue problems occur in numerous applications — e.g. see [7–14]. A backward error of an approximate eigenpair (x,λ) of a matrix A is a measure of the smallest perturbation E such that $(A+E)x = \lambda x$. This backward error can be used to determine if (x,λ) solves a nearby problem, by comparing the backward error with the size of any uncertainties in the data matrix A. A natural definition of the norm-wise backward error of an eigenpair (x,λ) is

$$\eta(x,\lambda) = \min\left\{\alpha^{-1}\|E\|_F : (A+E)x = \lambda x\right\} , \tag{1.1}$$

where α is a positive parameter that allows freedom in the way the perturbations are measured.

Let $X_k = (x_1, x_2, \cdots, x_k)$, $\Lambda_k = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_k)$, and $\{(x_j, \lambda_j),\ j = 1, \cdots, k\}$ be the set of approximate eigenpairs. In order to measure the backward error, the following definition given in Ref. [9] is a natural generalisation of the definition (1.1):

$$\eta(X_k, \Lambda_k) = \min\left\{\alpha^{-1}\|E\|_F : (A+E)X_k = X_k\Lambda_k\right\}.$$

Another definition for the backward error of eigenproblems for structured matrices is as follows. Let $\mathscr{K}$ be the set of some classes of structured matrices, and let

$$\eta_{\mathscr{K}}(X_k, \Lambda_k) = \min\left\{\alpha^{-1}\|E\|_F : (A+E)X_k = X_k\Lambda_k,\ A,\ A+E \in \mathscr{K}\right\} .$$

However, when $\mathscr{K} = \mathscr{K}_1,\ \mathscr{K}_2$ the backward errors for eigenproblem of these structured matrices have never been considered yet. We consider this problem, and present an explicit formula for $\eta_{\mathscr{K}_i}(X_k, \Lambda_k)$, $i = 1, 2$.

The remainder of this article is organised as follows. In Section 2, we present some useful lemmas to deduce our main results. In Section 3, computable backward errors $\eta_{\mathscr{K}_i}(X_k, \Lambda_k)$, $i = 1, 2$ are derived. Finally, some examples and our concluding remarks are given in Section 4.

2. Some Lemmas

We now present some lemmas, to be used in our subsequent derivation of structured backward errors.

Lemma 2.1 (cf. Ref. [15]). *Let* $Y,\ B\in\mathscr{C}^{m\times n}$ *be given and let*

$$\mathscr{L}=\{X\in\mathscr{C}^{m\times m}:XY=B,\ X^T=X\}\ .$$

Then $\mathscr{L}\neq\emptyset$ *if and only if* $BP_{Y^*}=B$ *and* $P_{\bar{Y}}BY^+=(P_{\bar{Y}}BY^+)^T$*; and if* $\mathscr{L}\neq\emptyset$*, then*

$$\begin{aligned}&\mathscr{L}'=\left\{BY^++(BY^+)^TP_Y^{\perp}+P_{\bar{Y}}^{\perp}HP_Y^{\perp}|H=H^T\right\}\ ,\\&\|X_{opt}\|_F=\min_{X\in\mathscr{L}}\|X\|_F\ ,\end{aligned}$$

where $X_{opt}=BY^++(BY^+)^TP_Y^{\perp}$.

Lemma 2.2 (cf. Ref. [15]). *Let* $X,\ B\in\mathscr{C}^{m\times k}$*,* $Y,\ C\in\mathscr{C}^{n\times k}$ *be given, and let*

$$S_k=\{A\in\mathscr{C}^{m\times n}:AY=B,\ A^TX=C\}\ .$$

Then

(i) $S_k\neq\emptyset$ *if and only if* $BP_{Y^*}=B,CP_{X^*}=C$ *and* $C^TY=X^TB$*; and*

(ii) if $S_k\neq\emptyset$*, then*

$$\begin{aligned}&S_k=\left\{BY^++(CX^+)^TP_Y^{\perp}+P_{\bar{X}}^{\perp}HP_Y^{\perp}|H\in\mathscr{C}^{m\times n}\right\}\ ,\\&\|A_{opt}\|_F=\min_{A\in S_k}\|A\|_F\ ,\end{aligned}$$

where $A_{opt}=BY^++(CX^+)^TP_Y^{\perp}$.

Lemma 2.3. *Let* $\mathscr{K}_1\subseteq\mathscr{C}^{m\times m}$ *be as given in Section* 1*, and let*

$$\begin{aligned}\Phi=&\left\{\begin{pmatrix}C_1 & D_1P_0\\ P_0^{-1}D_1 & P_0^{-1}C_1P_0\end{pmatrix}\in\mathscr{C}^{2k\times 2k},k\geq 1\right\}\\&\cup\left\{\begin{pmatrix}C_1 & \mu & D_1P_0\\ \mu^T & \beta & \mu^TP_0\\ P_0^{-1}D_1 & P_0^{-1}\mu & P_0^{-1}C_1P_0\end{pmatrix}\in\mathscr{C}^{(2k+1)\times(2k+1)},k\geq 0\right\}.\end{aligned}$$

Then $\mathscr{K}_1=\Phi$*, where* $C_1,D_1\in\mathscr{C}^{k\times k}$*,* $C_1=C_1^T$*,* $D_1=D_1^T$*,* $\mu\in\mathscr{C}^k$ *and* $\beta\in\mathscr{C}$*.*

Proof. Assuming that $A\in\mathscr{K}_1$, we have $A\in\mathscr{G}\mathscr{C}^{m\times m}$. Then from Definition 1.1, A has the following block forms:

for $m=2k$ $(k\geq 1)$,

$$A=\begin{pmatrix}C_1 & D_1P_0\\ P_0^{-1}D_1 & P_0^{-1}C_1P_0\end{pmatrix}\ ,$$

and for $m = 2k+1$ $(k \geq 0)$,

$$A = \begin{pmatrix} C_1 & \mu & D_1 P_0 \\ \nu^T & \beta & \nu^T P_0 \\ P_0^{-1} D_1 & P_0^{-1}\mu & P_0^{-1} C_1 P_0 \end{pmatrix},$$

where $C_1, D_1 \in \mathscr{C}^{k\times k}$, $\mu, \nu \in \mathscr{C}^k$, $\beta \in \mathscr{C}$.

Now since $A = A^T$, we have

$$\mu = \nu, \qquad C_1 = C_1^T, \qquad D_1 = D_1^T,$$

hence

$$A \in \Phi$$

and therefore

$$\mathscr{K}_1 \subseteq \Phi\,. \tag{2.1}$$

Conversely, if $A \in \Phi$ it follows that

$$A \in \mathscr{G}\mathscr{C}^{m\times m}$$

and

$$A = A^T,$$

hence

$$A \in \mathscr{K}_1$$

and therefore

$$\Phi \subseteq \mathscr{K}_1\,. \tag{2.2}$$

Consequently, from (2.1) and (2.2) we conclude that $\mathscr{K}_1 = \Phi$. □

Before we present the next lemma, let us introduce some notation — viz.

$$Q = \frac{1}{\sqrt{2}}\begin{pmatrix} I_k & -P_0 \\ I_k & P_0 \end{pmatrix} \in \mathscr{C}^{2k\times 2k}(k \geq 1)\,,$$

$$\tilde{Q} = \frac{1}{\sqrt{2}}\begin{pmatrix} I_k & 0 & -P_0 \\ 0^T & \sqrt{2} & 0^T \\ I_k & 0 & P_0 \end{pmatrix} \in \mathscr{C}^{(2k+1)\times(2k+1)}(k \geq 0)\,;$$

and for any matrices $Y, B \in \mathscr{C}^{m\times n}$,

$$QY = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \quad QB = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \qquad (m = 2k, k \geq 1)\,, \tag{2.3}$$

$$\tilde{Q}Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \quad \tilde{Q}B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \qquad (m = 2k+1, k \geq 0)\,, \tag{2.4}$$

where $Y_1, B_1 \in \mathscr{C}^{k\times n}$.

Lemma 2.4. *Given Y, $B \in \mathscr{C}^{m\times n}$ and Y_1, Y_2, B_1, B_2 as in* (2.3) *and* (2.4)*, for*

$$\mathscr{S}_{\mathscr{K}_1} = \{A \in \mathscr{K}_1 \subseteq \mathscr{C}^{m\times m} : AY = B\},$$
$$\tilde{\mathscr{S}}_{\mathscr{K}_1} = \{A \in \mathscr{C}^{k\times k} : AY_1 = B_1, A^T = A\},$$
$$\mathscr{S}'_{\mathscr{K}_1} = \{A \in \mathscr{C}^{(m-k)\times(m-k)} : AY_2 = B_2, A^T = A\},$$

we have $\mathscr{S}_{\mathscr{K}_1} \neq \emptyset$ if and only if $\tilde{\mathscr{S}}_{\mathscr{K}_1} \neq \emptyset$ and $\mathscr{S}'_{\mathscr{K}_1} \neq \emptyset$.

Proof. From Lemma 2.3, for any $A \in \mathscr{K}_1$, we have:
for $m = 2k (k \geq 1)$,

$$A = \begin{pmatrix} C_1 & D_1 P_0 \\ P_0^{-1} D_1 & P_0^{-1} C_1 P_0 \end{pmatrix},$$

and for $m = 2k+1 (k \geq 0)$,

$$A = \begin{pmatrix} C_1 & \mu & D_1 P_0 \\ \mu^T & \beta & \mu^T P_0 \\ P_0^{-1} D_1 & P_0^{-1}\mu & P_0^{-1} C_1 P_0 \end{pmatrix},$$

where $C_1, D_1 \in \mathscr{C}^{k\times k}$, $C_1 = C_1^T$, $D_1 = D_1^T$, $\mu \in \mathscr{C}^k$, $\beta \in \mathscr{C}$.
When $m = 2k$ $(k \geq 1)$, let $G = \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & I_k \\ -P_0^{-1} & P_0^{-1} \end{pmatrix}$ such that

$$G^T = G^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & -P_0 \\ I_k & P_0 \end{pmatrix}.$$

When $m = 2k+1$ $(k \geq 0)$, let $\tilde{G} = \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & 0 & I_k \\ 0^T & \sqrt{2} & 0^T \\ -P_0^{-1} & 0 & P_0^{-1} \end{pmatrix}$ such that

$$\tilde{G}^T = \tilde{G}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & 0 & -P_0 \\ 0^T & \sqrt{2} & 0^T \\ I_k & 0 & P_0 \end{pmatrix}.$$

It follows that for $m = 2k$,

$$G^T A G = \begin{pmatrix} C_1 - D_1 & O \\ O & C_1 + D_1 \end{pmatrix}, \tag{2.5}$$

and for $m = 2k+1$,

$$\tilde{G}^T A \tilde{G} = \begin{pmatrix} C_1 - D_1 & 0 & O \\ 0^T & \beta & \sqrt{2}\mu^T \\ O & \sqrt{2}\mu & C_1 + D_1 \end{pmatrix}. \tag{2.6}$$

Consequently, $AY = B$ is equivalent to

$$G^T AGG^T Y = G^T B, \tilde{G}^T A\tilde{G}\tilde{G}^T Y = \tilde{G}^T B\ . \tag{2.7}$$

Let $G^T Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ and $G^T B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ for $m = 2k$ and $\tilde{G}^T Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$, $\tilde{G}^T B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ for $m = 2k+1$, where $Y_1, B_1 \in \mathscr{C}^{k\times n}$.

Now assume that $\mathscr{S}_{\mathscr{K}_1} \neq \emptyset$, and let $A_0 \in \mathscr{S}_{\mathscr{K}_1}$. Since $A_0 \in \mathscr{K}_1$, from Lemma 2.3

$$A_0 = \begin{pmatrix} C_{10} & D_{10}P_0 \\ P_0^{-1}D_{10} & P_0^{-1}C_{10}P_0 \end{pmatrix} \qquad (m = 2k, k \geq 1)\,,$$

$$A_0 = \begin{pmatrix} C_{10} & \mu_0 & D_{10}P_0 \\ \mu_0^T & \beta_0 & \mu_0^T P_0 \\ P_0^{-1}D_{10} & P_0^{-1}\mu_0 & P_0^{-1}C_{10}P_0 \end{pmatrix} \qquad (m = 2k+1, k \geq 0)\,,$$

where $C_{10}, D_{10} \in \mathscr{C}^{k\times k}$, $C_{10} = C_{10}^T$, $D_{10} = D_{10}^T$, $\mu_0 \in \mathscr{C}^k$, $\beta_0 \in \mathscr{C}$.
From (2.5)-(2.7), for $m = 2k$ ($k \geq 1$) we have $C_{10} - D_{10} \in \tilde{\mathscr{S}}_{\mathscr{K}_1}$, $C_{10} + D_{10} \in \mathscr{S}'_{\mathscr{K}_1}$; and for $m = 2k+1$ ($k \geq 0$), we have $C_{10} - D_{10} \in \tilde{\mathscr{S}}_{\mathscr{K}_1}$, $\begin{pmatrix} \beta_0 & \sqrt{2}\mu_0^T \\ \sqrt{2}\mu_0 & C_{10} + D_{10} \end{pmatrix} \in \mathscr{S}'_{\mathscr{K}_1}$. It follows that $\tilde{\mathscr{S}}_{\mathscr{K}_1} \neq \emptyset$ and $\mathscr{S}'_{\mathscr{K}_1} \neq \emptyset$.

Conversely, suppose that $\tilde{\mathscr{S}}_{\mathscr{K}_1} \neq \emptyset$ and $\mathscr{S}'_{\mathscr{K}_1} \neq \emptyset$, and also $H_0 \in \tilde{\mathscr{S}}_{\mathscr{K}_1}$ and $Z_0 \in \mathscr{S}'_{\mathscr{K}_1}$. Then for $m = 2k$ ($k \geq 1$), it is easy to verify that

$$\tilde{A}_0 = \begin{pmatrix} \dfrac{(H_0 + Z_0)}{2} & \dfrac{(Z_0 - H_0)}{2}P_0 \\ P_0^{-1}\dfrac{(Z_0 - H_0)}{2} & P_0^{-1}\dfrac{(H_0 + Z_0)}{2}P_0 \end{pmatrix} \in \mathscr{S}_{\mathscr{K}_1}\,.$$

For $m = 2k+1$ ($k \geq 0$), let $Z_0 = \begin{pmatrix} z & Z_{01}^T \\ Z_{01} & Z_{02} \end{pmatrix}$, where $z \in \mathscr{C}$, $Z_{01} \in \mathscr{C}^k$, $Z_{02} \in \mathscr{C}^{k\times k}$ and $Z_{02}^T = Z_{02}$. It then follows that

$$\tilde{A}_0 = \begin{pmatrix} \dfrac{(H_0 + Z_{02})}{2} & \dfrac{\sqrt{2}}{2}Z_{01} & \dfrac{(Z_{02} - H_0)}{2}P_0 \\ \dfrac{\sqrt{2}}{2}Z_{01}^T & z & \dfrac{\sqrt{2}}{2}Z_{01}^T P_0 \\ P_0^{-1}\dfrac{(Z_{02} - H_0)}{2} & \dfrac{\sqrt{2}}{2}P_0^{-1}Z_{01} & P_0^{-1}\dfrac{(H_0 + Z_{02})}{2}P_0 \end{pmatrix} \in \mathscr{S}_{\mathscr{K}_1}\,.$$

Hence $\mathscr{S}_{\mathscr{K}_1} \neq \emptyset$, which implies the desired result. □

Lemma 2.5. *Let* $\mathscr{S}_{\mathscr{K}_1}$*,* Y_i*,* B_i *be given as in Lemma* 2.4*. Then* $\mathscr{S}_{\mathscr{K}_1} \neq \emptyset$ *if and only if* $B_i P_{Y_i^*} = B_i$ *and* $P_{\bar{Y}_i} B_i Y_i^+ = (P_{\bar{Y}_i} B_i Y_i^+)^T$*,* $i = 1, 2$*.*

Proof. With $\tilde{\mathscr{S}}_{\mathscr{K}_1}, \mathscr{S}'_{\mathscr{K}_1}$ given as in Lemma 2.4, from Lemma 2.1 we have $\tilde{\mathscr{S}}_{\mathscr{K}_1} \neq \emptyset$ and $\mathscr{S}'_{\mathscr{K}_1} \neq \emptyset$ if and only if $B_i P_{Y_i^*} = B_i$ and $P_{\bar{Y}_i} B_i Y_i^+ = (P_{\bar{Y}_i} B_i Y_i^+)^T$, $i = 1, 2$. Thus from Lemma 2.4 we have the desired result. □

Lemma 2.6. *Let* $\mathscr{K}_2 \subseteq \mathscr{C}^{m\times m}$ *be as given in Section* 1*, and let*

$$\Psi = \left\{ \begin{pmatrix} C_1 & D_1 P_0 \\ -P_0^{-1} D_1 & -P_0^{-1} C_1 P_0 \end{pmatrix} \in \mathscr{C}^{2k\times 2k}, k \geq 1 \right\}$$
$$\cup \left\{ \begin{pmatrix} C_1 & \mu & D_1 P_0 \\ -\mu^T & 0 & \mu^T P_0 \\ -P_0^{-1} D_1 & -P_0^{-1}\mu & -P_0^{-1} C_1 P_0 \end{pmatrix} \in \mathscr{C}^{(2k+1)\times(2k+1)}, k \geq 0 \right\} .$$

Then $\mathscr{K}_2 = \Psi$*, where* $C_1, D_1 \in \mathscr{C}^{k\times k}$*,* $C_1 = -C_1^T$*,* $D_1 = D_1^T$ *and* $\mu \in \mathscr{C}^k$*.*

Proof. For $A \in \mathscr{K}_2$, we have $A \in \mathscr{G}\tilde{\mathscr{C}}^{m\times m}$. Then from Definition 1.1, A has the following block forms:

for $m = 2k$ $(k \geq 1)$,

$$A = \begin{pmatrix} C_1 & D_1 P_0 \\ -P_0^{-1} D_1 & -P_0^{-1} C_1 P_0 \end{pmatrix} ;$$

and for $m = 2k+1$ $(k \geq 0)$,

$$A = \begin{pmatrix} C_1 & \mu & D_1 P_0 \\ -\nu^T & \beta & \nu^T P_0 \\ -P_0^{-1} D_1 & -P_0^{-1}\mu & -P_0^{-1} C_1 P_0 \end{pmatrix} ,$$

where $C_1, D_1 \in \mathscr{C}^{k\times k}$, $\mu, \nu \in \mathscr{C}^k$ and $\beta \in \mathscr{C}$.

Since $A = -A^T$, we have $\mu = \nu, C_1 = -C_1^T, D_1 = D_1^T$ and $\beta = 0$ such that $A \in \Psi$, and hence

$$\mathscr{K}_2 \subseteq \Psi . \tag{2.8}$$

Conversely, when $A \in \Psi$ it follows that $A \in \mathscr{G}\tilde{\mathscr{C}}^{m\times m}$ and $A = -A^T$ such that $A \in \mathscr{K}_2$, and hence

$$\Psi \subseteq \mathscr{K}_2 . \tag{2.9}$$

Combining (2.8) and (2.9), we therefore have $\mathscr{K}_2 = \Psi$. □

Lemma 2.7. *Given* Y*,* $B \in \mathscr{C}^{m\times n}$ *and* Y_1*,* Y_2*,* B_1*,* B_2 *as in* (2.3) and (2.4)*, and letting*

$$\mathscr{S}_{\mathscr{K}_2} = \{A \in \mathscr{K}_2 \subseteq \mathscr{C}^{m\times m} : AY = B\} ,$$
$$\tilde{\mathscr{S}}_{\mathscr{K}_2} = \{A \in \mathscr{C}^{k\times(m-k)} : AY_2 = B_1, A^T Y_1 = -B_2\} ,$$

we have that $\mathscr{S}_{\mathscr{K}_2} \neq \emptyset$ *if and only if* $\tilde{\mathscr{S}}_{\mathscr{K}_2} \neq \emptyset$*.*

Proof. From Lemma 2.6, for any $A \in \mathscr{K}_2$:

for $m = 2k$ $(k \geq 1)$,

$$A = \begin{pmatrix} C_1 & D_1 P_0 \\ -P_0^{-1} D_1 & -P_0^{-1} C_1 P_0 \end{pmatrix},$$

and for $m = 2k+1$ $(k \geq 0)$,

$$A = \begin{pmatrix} C_1 & \mu & D_1 P_0 \\ -\mu^T & 0 & \mu^T P_0 \\ -P_0^{-1} D_1 & -P_0^{-1}\mu & -P_0^{-1} C_1 P_0 \end{pmatrix},$$

where $C_1, D_1 \in \mathscr{C}^{k\times k}$, $C_1 = -C_1^T$, $D_1 = D_1^T$, $\mu \in \mathscr{C}^k$.

Let $G, \tilde{G}$ be as given in Lemma 2.4, such that

$$\begin{aligned} G^T A G &= \begin{pmatrix} O & C_1 + D_1 \\ C_1 - D_1 & O \end{pmatrix} = \begin{pmatrix} O & C_1 + D_1 \\ -(C_1 + D_1)^T & O \end{pmatrix} \\ &= \begin{pmatrix} O & L \\ -L^T & O \end{pmatrix}, \end{aligned} \tag{2.10}$$

$$\tilde{G}^T A \tilde{G} = \begin{pmatrix} O & \sqrt{2}\mu & C_1 + D_1 \\ -\sqrt{2}\mu^T & 0 & 0^T \\ C_1 - D_1 & 0 & O \end{pmatrix} = \begin{pmatrix} O & \tilde{L} \\ -\tilde{L}^T & O \end{pmatrix}, \tag{2.11}$$

where $L = C_1 + D_1$, $\tilde{L} = (\sqrt{2}\mu, C_1 + D_1)$.

Also $AY = B$ is equivalent to

$$G^T A G G^T Y = G^T B, \tilde{G}^T A \tilde{G} \tilde{G}^T Y = \tilde{G}^T B. \tag{2.12}$$

Assuming that $\mathscr{S}_{\mathscr{K}_2} \neq \emptyset$, we let $A_0 \in \mathscr{S}_{\mathscr{K}_2}$. Since $A_0 \in \mathscr{K}_2$, from Lemma 2.6

$$A_0 = \begin{pmatrix} C_{10} & D_{10} P_0 \\ -P_0^{-1} D_{10} & -P_0^{-1} C_{10} P_0 \end{pmatrix} \qquad (m = 2k,\ k \geq 1),$$

$$A_0 = \begin{pmatrix} C_{10} & \mu_0 & D_{10} P_0 \\ -\mu_0^T & 0 & \mu_0^T P_0 \\ -P_0^{-1} D_{10} & -P_0^{-1}\mu_0 & -P_0^{-1} C_{10} P_0 \end{pmatrix} (m = 2k+1,\ k \geq 0),$$

where $C_{10}, D_{10} \in \mathscr{C}^{k\times k}$, $C_{10} = -C_{10}^T$, $D_{10} = D_{10}^T$, $\mu_0 \in \mathscr{C}^k$. From (2.10)-(2.12), for $m = 2k$ $(k \geq 1)$ we have $C_{10} + D_{10} \in \tilde{\mathscr{S}}_{\mathscr{K}_2}$; and for $m = 2k+1$ $(k \geq 0)$ we have $(\sqrt{2}\mu_0, C_{10} + D_{10}) \in \tilde{\mathscr{S}}_{\mathscr{K}_2}$, which implies that $\tilde{\mathscr{S}}_{\mathscr{K}_2} \neq \emptyset$.

Conversely, assume that $\tilde{\mathscr{S}}_{\mathscr{K}_2} \neq \emptyset$. Suppose $H_0 \in \tilde{\mathscr{S}}_{\mathscr{K}_2}$, when for $m = 2k$ $(k \geq 1)$ it is easy to verify that

$$\tilde{A}_0 = \begin{pmatrix} \dfrac{(H_0 - H_0^T)}{2} & \dfrac{(H_0 + H_0^T)}{2} P_0 \\ -P_0^{-1} \dfrac{(H_0 + H_0^T)}{2} & -P_0^{-1} \dfrac{(H_0 - H_0^T)}{2} P_0 \end{pmatrix} \in \mathscr{S}_{\mathscr{K}_2}.$$

For $m=2k+1$ ($k\geq 0$), we let $H_0=\begin{pmatrix} h & H_{01}\end{pmatrix}$ where $h\in\mathscr{C}^k$ and $H_{01}\in\mathscr{C}^{k\times k}$. Then

$$\tilde{A}_0=\begin{pmatrix} \dfrac{(H_{01}-H_{01}^T)}{2} & \dfrac{\sqrt{2}}{2}h & \dfrac{(H_{01}+H_{01}^T)}{2}P_0 \\ \dfrac{\sqrt{2}}{2}h^T & 0 & \dfrac{\sqrt{2}}{2}h^T P_0 \\ -P_0^{-1}\dfrac{(H_{01}+H_{01}^T)}{2} & -\dfrac{\sqrt{2}}{2}P_0^{-1}h & -P_0^{-1}\dfrac{(H_{01}-H_{01}^T)}{2}P_0 \end{pmatrix}\in\mathscr{S}_{\mathscr{K}_2}\,.$$

Hence $\mathscr{S}_{\mathscr{K}_2}\neq\emptyset$, which implies the desired result. ☐

Lemma 2.8. *Let $\mathscr{S}_{\mathscr{K}_2},Y_i,B_i$, $i=1,2$ be as given in Lemma 2.7. Then $\mathscr{S}_{\mathscr{K}_2}\neq\emptyset$ if and only if $B_1P_{Y_2^*}=B_1$, $B_2P_{Y_1^*}=B_2$ and $B_1^TY_1=-Y_2^TB_2$.*

Proof. The result follows from Lemmas 2.2 and 2.7. ☐

3. Backward Errors

Some explicit formulae for the structured backward error $\eta_{\mathscr{K}_i}(X_k,\Lambda_k)$, $i=1,2$ are now derived.

Let $A\in\mathscr{C}^{m\times m}$, $X_k\in\mathscr{C}^{m\times k}$, $\Lambda_k=\mathrm{diag}(\lambda_1,\cdots,\lambda_k)$, $m\geq k$, and $\mathscr{K}_i$ be given as in Section 1, and denote $\mathbf{S}_{\mathscr{K}_i}=\{A+E\in\mathscr{K}_i:(A+E)X_k=X_k\Lambda_k\}$, $i=1,2$. When $m=2k$ ($k\geq 1$) and Q is given in (2.3) let

$$QX_k=\begin{pmatrix} X_1\\ X_2\end{pmatrix},\quad QX_k\Lambda_k=\begin{pmatrix} T_1\\ T_2\end{pmatrix},\quad Q(X_k\Lambda_k-AX_k)=\begin{pmatrix} F_1\\ F_2\end{pmatrix},$$

and when $m=2k+1(k\geq 0)$ and $\tilde{Q}$ is given in (2.4) let

$$\tilde{Q}X_k=\begin{pmatrix} X_1\\ X_2\end{pmatrix},\quad \tilde{Q}X_k\Lambda_k=\begin{pmatrix} T_1\\ T_2\end{pmatrix},\quad \tilde{Q}(X_k\Lambda_k-AX_k)=\begin{pmatrix} F_1\\ F_2\end{pmatrix},$$

where $X_1,T_1,F_1\in\mathscr{C}^{k\times k}$. To obtain $\eta_{\mathscr{K}_i}(X_k,\Lambda_k)$, one needs to assume that $\mathbf{S}_{\mathscr{K}_i}\neq\emptyset$, $i=1,2$, so we first provide some necessary and sufficient conditions (or just sufficient conditions) for $\mathbf{S}_{\mathscr{K}_i}\neq\emptyset$.

Lemma 3.1. $\mathbf{S}_{\mathscr{K}_1}\neq\emptyset$ *if and only if*

$$T_iP_{X_i^*}=T_i\,, \tag{3.1}$$

$$\text{and}\quad P_{\bar{X}_i}T_iX_i^+=(P_{\bar{X}_i}T_iX_i^+)^T\,,\quad i=1,2\,. \tag{3.2}$$

Proof. From Lemma 2.5, we have $\mathbf{S}_{\mathscr{K}_1}\neq\emptyset$ if and only if (3.1) and (3.2) hold. ☐

Lemma 3.2. $\mathbf{S}_{\mathscr{K}_2}\neq\emptyset$ *if and only if $T_1^TX_1=-X_2^TT_2$, $T_1P_{X_2^*}=T_1$ and $T_2P_{X_1^*}=T_2$.*

Proof. The result follows from Lemma 2.8. □

The following theorem provides an explicit formula for $\eta_{\mathscr{K}_1}(X_k, \Lambda_k)$.

Theorem 3.1. *Let $A \in \mathscr{K}_1 \subseteq \mathscr{C}^{m\times m}$, $X_k \in \mathscr{C}^{m\times k}$, $\Lambda_k \in \mathscr{C}^{k\times k}$, where $m \geq k$. Assume that* (3.1) *and* (3.2) *are satisfied. Then for any $\alpha > 0$,*

$$\eta_{\mathscr{K}_1}(X_k, \Lambda_k) = \alpha^{-1}\left[\sum_{i=1,2}\left\|F_i X_i^+ + (F_i X_i^+)^T P_{X_i}^{\perp}\right\|_F^2\right]^{\frac{1}{2}}. \tag{3.3}$$

Proof. From Lemma 3.1, $\mathbf{S}_{\mathscr{K}_1} = \{A+E \in \mathscr{K}_1 : (A+E)X_k = X_k\Lambda_k\} \neq \emptyset$. Since $A+E, A \in \mathscr{K}_1$, from Lemma 2.3 it is also easy to see that $E \in \mathscr{K}_1$.

Now let E have the following block forms:
for $m = 2k$ $(k \geq 1)$,

$$E = \begin{pmatrix} E_1 & E_2 P_0 \\ P_0^{-1}E_2 & P_0^{-1}E_1 P_0 \end{pmatrix}$$

when

$$QEQ^T = \begin{pmatrix} E_1 - E_2 & O \\ O & E_1 + E_2 \end{pmatrix}, \tag{3.4}$$

and for $m = 2k+1 (k \geq 0)$,

$$E = \begin{pmatrix} E_1 & \mu & E_2 P_0 \\ \mu^T & \beta & \mu^T P_0 \\ P_0^{-1}E_2 & P_0^{-1}\mu & P_0^{-1}E_1 P_0 \end{pmatrix}$$

when

$$\tilde{Q}E\tilde{Q}^T = \begin{pmatrix} E_1 - E_2 & 0 & O \\ 0^T & \beta & \sqrt{2}\mu^T \\ O & \sqrt{2}\mu & E_1 + E_2 \end{pmatrix} = \begin{pmatrix} E_1 - E_2 & O \\ O & J \end{pmatrix}, \tag{3.5}$$

where $E_1, E_2 \in \mathscr{C}^{k\times k}$, $\mu \in \mathscr{C}^k$, $\beta \in \mathscr{C}$, $E_1 = E_1^T$, $E_2 = E_2^T$, $J = \begin{pmatrix} \beta & \sqrt{2}\mu^T \\ \sqrt{2}\mu & E_1 + E_2 \end{pmatrix}$.

Since $EX_k = X_k\Lambda_k - AX_k$ is equivalent to

$$\begin{aligned} &QEQ^T QX_k = Q(X_k\Lambda_k - AX_k) \quad (m = 2k,\ k \geq 1)\,, \\ &\tilde{Q}E\tilde{Q}^T \tilde{Q}X_k = \tilde{Q}(X_k\Lambda_k - AX_k) \quad (m = 2k+1,\ k \geq 0)\,, \end{aligned}$$

it follows from (3.4) and (3.5) that

$$(E_1 - E_2)X_1 = F_1, \quad (E_1 + E_2)X_2 = F_2, \quad (m = 2k,\ k \geq 1),$$

and

$$(E_1 - E_2)X_1 = F_1, \quad JX_2 = F_2, \qquad (m = 2k+1,\ k \geq 0)\,,$$

where $E_1 - E_2 = (E_1 - E_2)^T$, $E_1 + E_2 = (E_1 + E_2)^T$, $J = J^T$.

From Lemma 2.1, when $E_1 - E_2 = F_1X_1^+ + (F_1X_1^+)^T P_{X_1}^\perp$ we have that $\|E_1 - E_2\|_F$ is minimised. For $m = 2k$ ($k \geq 1$), when $E_1 + E_2 = F_2X_2^+ + (F_2X_2^+)^T P_{X_2}^\perp$ we have that $\|E_1 + E_2\|_F$ is minimised. For $m = 2k+1$ ($k \geq 0$), when $J = F_2X_2^+ + (F_2X_2^+)^T P_{X_2}^\perp$ we have that $\|J\|_F$ is minimised. Consequently, either

$$\begin{aligned}\|E\|_F =& \|QEQ^T\|_F = \left(\|E_1 - E_2\|_F^2 + \|E_1 + E_2\|_F^2\right)^{\frac{1}{2}} \\ =& \left[\sum_{i=1,2} \left\|F_iX_i^+ + (F_iX_i^+)^T P_{X_i}^\perp\right\|_F^2\right]^{\frac{1}{2}} \quad (m = 2k,\ k \geq 1)\end{aligned}$$

or

$$\begin{aligned}\|E\|_F =& \|\tilde{Q}E\tilde{Q}^T\|_F = \left(\|E_1 - E_2\|_F^2 + \|J\|_F^2\right)^{\frac{1}{2}} \\ =& \left[\sum_{i=1,2} \left\|F_iX_i^+ + (F_iX_i^+)^T P_{X_i}^\perp\right\|_F^2\right]^{\frac{1}{2}} \quad (m = 2k+1, k \geq 0)\end{aligned}$$

is minimised, and therefore

$$\eta_{\mathscr{K}_1}(X_k, \Lambda_k) = \alpha^{-1}\left[\sum_{i=1,2} \left\|F_iX_i^+ + (F_iX_i^+)^T P_{X_i}^\perp\right\|_F^2\right]^{\frac{1}{2}}.$$

□

The following theorem provides an explicit formula of $\eta_{\mathscr{K}_2}(X_k, \Lambda_k)$.

Theorem 3.2. *Let $A \in \mathscr{K}_2 \subseteq \mathscr{C}^{m\times m}$, $X_k \in \mathscr{C}^{m\times k}$, $\Lambda_k \in \mathscr{C}^{k\times k}$, $m \geq k$. Assume that $T_1^T X_1 = -X_2^T T_2$, $T_1 P_{X_2^*} = T_1$, $T_2 P_{X_1^*} = T_2$. Then for any $\alpha > 0$*

$$\eta_{\mathscr{K}_2}(X_k, \Lambda_k) = \frac{\sqrt{2}}{\alpha}\left\|F_1X_2^+ - (F_2X_1^+)^T P_{X_2}^\perp\right\|_F. \tag{3.6}$$

Proof. From Lemma 3.2, $\mathbf{S}_{\mathscr{K}_2} = \{A+E \in \mathscr{K}_2 : (A+E)X_k = X_k\Lambda_k\} \neq \emptyset$. Since $A+E, A \in \mathscr{K}_2$, from Lemma 2.6 we have $E \in \mathscr{K}_2$.

Let E have the following block form:
for $m = 2k$ ($k \geq 1$),

$$E = \begin{pmatrix} E_1 & E_2P_0 \\ -P_0^{-1}E_2 & -P_0^{-1}E_1P_0 \end{pmatrix}$$

when

$$QEQ^T = \begin{pmatrix} O & E_1+E_2 \\ E_1-E_2 & O \end{pmatrix} = \begin{pmatrix} O & E_1+E_2 \\ -(E_1+E_2)^T & O \end{pmatrix}, \tag{3.7}$$

and for $m = 2k+1$ ($k \geq 0$),

$$E = \begin{pmatrix} E_1 & \mu & E_2 P_0 \\ -\mu^T & 0 & \mu^T P_0 \\ -P_0^{-1} E_2 & -P_0^{-1}\mu & -P_0^{-1} E_1 P_0 \end{pmatrix}$$

when

$$\tilde{Q}E\tilde{Q}^T = \begin{pmatrix} O & \sqrt{2}\mu & E_1+E_2 \\ -\sqrt{2}\mu^T & 0 & 0^T \\ E_1-E_2 & 0 & O \end{pmatrix} = \begin{pmatrix} O & N \\ -N^T & O \end{pmatrix}, \tag{3.8}$$

where $E_1, E_2 \in \mathscr{C}^{k\times k}$, $\mu \in \mathscr{C}^k$, $E_1^T = -E_1$, $E_2^T = E_2$ and $N = (\sqrt{2}\mu, E_1+E_2)$.

Since

$$QEQ^T QX_k = Q(X_k\Lambda_k - AX_k) \quad (m = 2k, k \geq 1),$$

and

$$\tilde{Q}E\tilde{Q}^T\tilde{Q}X_k = \tilde{Q}(X_k\Lambda_k - AX_k) \quad (m = 2k+1, k \geq 0),$$

it follows from (3.7) and (3.8) that

$$(E_1+E_2)X_2 = F_1, \quad (E_1+E_2)^T X_1 = -F_2, \quad (m = 2k, k \geq 1)$$

and

$$NX_2 = F_1, \quad N^T X_1 = -F_2, \quad (m = 2k+1, k \geq 0).$$

From Lemma 2.2, when $E_1 + E_2 = F_1X_2^+ - (F_2X_1^+)^T P_{X_2}^\perp$ we have $\|E_1+E_2\|_F$ minimised; and when $N = F_1X_2^+ - (F_2X_1^+)^T P_{X_2}^\perp$, we have $\|N\|_F$ minimised. Consequently, we have $\|E\|_F = \|QEQ^T\|_F = \sqrt{2}\|E_1+E_2\|_F = \sqrt{2}\|F_1X_2^+ - (F_2X_1^+)^T P_{X_2}^\perp\|_F$ ($m = 2k,\ k \geq 1$) or $\|E\|_F = \|\tilde{Q}E\tilde{Q}^T\|_F = \sqrt{2}\|N\|_F = \sqrt{2}\|F_1X_2^+ - (F_2X_1^+)^T P_{X_2}^\perp\|_F$ ($m = 2k+1,\ k \geq 0$) minimised, and therefore

$$\eta_{\mathscr{K}_2}(X_k, \Lambda_k) = \frac{\sqrt{2}}{\alpha} \left\| F_1X_2^+ - (F_2X_1^+)^T P_{X_2}^\perp \right\|_F .$$

□

4. Examples and Remarks

In this section, we give two examples to compute the backward errors $\eta_{\mathscr{K}_1}(X_k,\Lambda_k)$ and $\eta_{\mathscr{K}_2}(X_k,\Lambda_k)$.

Example 4.1. Consider $P_0=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}$, when in (2.3) $Q=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & -1\\ 0 & 1 & -1 & 0\\ 1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0\end{pmatrix}$. Let $A=\begin{pmatrix} 0 & 1 & -i & 1\\ -1 & 0 & 1 & 1\\ i & -1 & 0 & -1\\ -1 & -1 & 1 & 0\end{pmatrix}\in\mathscr{K}_2\subseteq\mathscr{C}^{4\times4}$, $X_k=\frac{1}{\sqrt{2}}\begin{pmatrix} 1+i & 2+2i\\ 1+i & -1-i\\ 1-i & -1+i\\ 1-i & 2-2i\end{pmatrix}\in\mathscr{C}^{4\times2}$, $\Lambda_k=\begin{pmatrix} 2 & 0\\ 0 & 2\end{pmatrix}\in\mathscr{C}^{2\times2}$. Then a simple calculation yields

$$QX_k=\begin{pmatrix} i & 2i\\ i & -i\\ 1 & 2\\ 1 & -1\end{pmatrix}:=\begin{pmatrix} X_1\\ X_2\end{pmatrix},$$

$$QX_k\Lambda_k=\begin{pmatrix} 2i & 4i\\ 2i & -2i\\ 2 & 4\\ 2 & -2\end{pmatrix}:=\begin{pmatrix} T_1\\ T_2\end{pmatrix},$$

where $X_1=\begin{pmatrix} i & 2i\\ i & -i\end{pmatrix}$, $T_1=\begin{pmatrix} 2i & 4i\\ 2i & -2i\end{pmatrix}$. It is easy to verify that

$$T_1^TX_1=-X_2^TT_2=\begin{pmatrix} -4 & -2\\ -2 & -10\end{pmatrix},$$

and a simple calculation gives

$$Q(X_k\Lambda_k-AX_k)=\frac{1}{\sqrt{2}}\begin{pmatrix} -2+2i & -4+10i\\ -4+8i & -2+4i\\ 4+4i & 8+2i\\ 6+2i & -2i\end{pmatrix}:=\begin{pmatrix} F_1\\ F_2\end{pmatrix},$$

where $F_1=\frac{1}{\sqrt{2}}\begin{pmatrix} -2+2i & -4+10i\\ -4+8i & -2+4i\end{pmatrix}$. Hence from Theorem 3.2,

$$\eta_{\mathscr{K}_2}(X_k,\Lambda_k)=\frac{8.0001}{\alpha}.$$

Example 4.2. Consider $P_0 = \begin{pmatrix} 0 & 0 & \cdots & 1 \\ 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & \cdots & 0 \end{pmatrix} \subseteq \mathscr{C}^{2500\times 2500}$,

$$C = diag\left(\begin{pmatrix} -1 & i \\ i & 2 \end{pmatrix}, \begin{pmatrix} -1 & i \\ i & 2 \end{pmatrix}, \cdots, \begin{pmatrix} -1 & i \\ i & 2 \end{pmatrix}\right) \subseteq \mathscr{C}^{2500\times 2500},$$

$$D = diag\left(\begin{pmatrix} i & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} i & 1 \\ 1 & 2 \end{pmatrix}, \cdots, \begin{pmatrix} i & 1 \\ 1 & 2 \end{pmatrix}\right) \subseteq \mathscr{C}^{2500\times 2500},$$

$$A = \begin{pmatrix} C & DP_0 \\ P_0^{-1}D & P_0^{-1}CP_0 \end{pmatrix} \in \mathscr{K}_1 \subseteq \mathscr{C}^{5000\times 5000}.$$

From Theorem 3.1, we compute

$$\eta_{\mathscr{K}_1}(X_k, \Lambda_k) = \frac{8.4092}{\alpha}.$$

When A is some stochastic symmetric generalised centrosymmetric matrices or skew-symmetric generalised skew- centrosymmetric matrices, by simple calculations one sees that the above inequalities still hold, implying both structured stability and stability of the numerical algorithm. For the backward errors for the eigenproblem of a special class of symmetric generalised centrosymmetric and skew-symmetric generalised skew-centrosymmetric matrices, the corresponding explicit formulae are in (3.3) and (3.6)). In particular, when the orthogonal matrix P_0 reduces to a subidentity matrix, (3.3) and (3.6) reduce to the explicit formulae of backward errors for the eigenproblem of symmetric centrosymmetric and skew-symmetric skew-centrosymmetric matrices [6].

Acknowledgments

The work was supported by the Natural Science Foundation for Colleges and Universities of Jiangsu Province of China under Grant No. 13KJB110020 and the Jiangsu Provincial Natural Science Foundation of Jiangsu Province of China under Grant No. BK20130985.

References

[1] Z. Xu, K.Y. Zhang and Q. Lu, *Fast Algorithms of Toeplitz Form*, Northwest Industry University Press (1999).
[2] I.J. Good, *The inverse of an entro-symmetric matrix*, Technometrics, **12**, 153-156 (1970).
[3] W.C. Pye, T.L. Boullion and T.A. Atchison, *The pseudoinverse of s centro-symmetric matrix*, Linear Algebra Appl. **6**, 201-204 (1973).
[4] A. Andrew, *Centrosymmetric matrices*, SIAM Review **40**, 697-698 (1998).
[5] A. Cantoni and P. Butler, *Eigenvalues and eigenvectors of symmetric centrosymmetric matrices*, Linear Algebra Appl. **13**, 275-288 (1976).

[6] D. Tao, M. Yasuda, *A spectral characterization of generalized real symmetric centrosymmetric and generalized real symmetric skew-centrosymmetric matrices*, SIAM J. Matrix Anal. Appl. **23**, 885-895 (2002).

[7] J.-G. Sun, *Backward errors for the unitary eigenproblem*, Technical Report UMINF-97.25, Department of Computing Science, University of Umeå, Sweden (1997).

[8] A. Bunse-Gerstner, R. Byers and V. Mehrmann, *A chart of numerical methods for structured eigenvalue problems*, SIAM J. Matrix Anal. Appl. **13**, 419-453 (1992).

[9] N.J. Higham, *Accuracy and Stability of Numerical Algorithms*, SIAM Press (2002).

[10] R.C. Li, *Relative perturbation theory: I eigenvalue and singular value variations*, SIAM J. Matrix Anal. Appl. **19**, 956-982 (1998).

[11] C.K. Li, R.C. Li and Q. Ye *Eigenvalues of an alignment matrix in nonlinear manifold learning*, Comm. Math. Sc. **5**, 313-329 (2007).

[12] R.C. Li, *Relative perturbation theory:(III) more bounds on eigenvalue variation*, Linear Algebra Appl. **266**, 337-345 (1997).

[13] Z.-J. Bai, *Error analysis of Lanczos algorithm for nonsymmetric eigenvalue problem*, Math. Comp. bf 62, 209-226 (1994).

[14] Z.-J. Bai, D. Day and Q. Ye *ABLE: An adaptive block Lanczos method for non-Hermitian eigenvalue problems*, SIAM J. Matrix Anal. Appl. **20**, 1060-1082 (1999).

[15] M.-S. Wei, *Theory and Computations for Generalised Least Squares Problems* (in Chinese), Science Press, Beijing (2006).

[16] N. Sugiyama, *Derivation of system matrices from nonlinear dynamic simulation of jet engines*, Journal of Guidance, Control, and Dynamics. **17**, 1320-1326 (1994).

[17] N. Sugiyama, *System identification of jet engines*, J. Eng. Gas Turbines Power. **122**, 19-26 (1999).

[18] E. Naderi, N. Meskin and K. Khorasani, *Nonlinear fault diagnosis of jet engines by using a multiple model-based approach*, J. Eng. Gas Turbines Power. **134**, 1-10 (2011).