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An Explicit Second-Order Numerical Scheme to Solve Decoupled Forward Backward Stochastic Equations

Yu Fu and Weidong Zhao*

School of Mathematics & Finance Institute, Shandong University, Jinan 250100, China.

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Abstract. An explicit numerical scheme is proposed for solving decoupled forward backward stochastic differential equations (FBSDE) represented in integral equation form. A general error inequality is derived for this numerical scheme, which also implies its stability. Error estimates are given based on this inequality, showing that the explicit scheme can be second-order. Some numerical experiments are carried out to illustrate the high accuracy of the proposed scheme.

AMS subject classifications: 60H35, 65C20, 60H10

Key words: Explicit scheme, second-order, decoupled FBSDE, error estimate.

1. Introduction

We consider the numerical solution of decoupled forward backward stochastic differential equations (FBSDE) on a filtered complete probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, represented in the following equivalent integral equation form:

$$\begin{cases} X_t = X_0 + \displaystyle\int_0^t b(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_s)\mathrm{d}W_s\,, \quad t \in [0, T]\,, \quad \text{(SDE)} \\ Y_t = \xi + \displaystyle\int_t^T f(s, X_s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s\,, \quad t \in [0, T]\,, \quad \text{(BSDE)} \end{cases} \tag{1.1}$$

where $\mathbb{F} = (\mathscr{F}_t)_{0\le t\le T}$ is the natural filtration of the standard d-dimensional Brownian motion $W = (W_t)_{0\le t\le T}$, $\mathscr{F} = \mathscr{F}_T$ with the fixed finite horizon T, $\xi \in \mathscr{F}_T$ is an L^2 integrable random variable, $b\colon \Omega \times [0, T] \times \mathbb{R}^q \to \mathbb{R}^q$, $\sigma\colon \Omega \times [0, T] \times \mathbb{R}^q \to \mathbb{R}^{q\times d}$ and $f\colon \Omega \times [0, T] \times \mathbb{R}^q \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, are all measurable functions, and $b(t, x)$, $\sigma(t, x)$ and $f(t, x, y, z)$ are $\mathscr{F}_t$-measurable for fixed $(X_t, Y_t, Z_t) = (x, y, z)$. Note that the integrals in (1.1) with respect

*Corresponding author. *Email addresses:* nielf0614@126.com (Yu Fu), wdzhao@sdu.edu.cn (Weidong Zhao)

to the Brownian motion W_s are of the Itô type, and the first equation arises from a standard forward stochastic differential equation (SDE), while the second arises from a backward stochastic differential equation (BSDE) since the terminal condition of $Y_T = \xi$ is given. A triple (X_t, Y_t, Z_t) is called an L^2 solution of decoupled FBSDE (1.1) if it is $\mathscr{F}_t$-adapted and square integrable.

In 1990, Pardoux and Peng proved the existence and uniqueness of the solution for nonlinear BSDE in their original work [13]. There has since been increasing attention paid to the theory of FBSDE and their application in many fields such as mathematical finance, partial differential equations (PDE), stochastic control, risk measure and game theory. However, the solution of FBSDE in closed form can seldom be found in practical problems, so numerical methods are often invoked. Under certain conditions, the relationship between the solutions of the decoupled FBSDE represented by (1.1) and parabolic PDE has led to some numerical methods to solve the FBSDE based on solving the corresponding parabolic PDE [3,7–9,11]. There are also other numerical schemes directly developed from the FBSDE [1,4,15–18].

Nevertheless, most existing highly accurate numerical methods to solve for Y_t are implicit, with a heavy computational requirement. By using the properties of the Itô integral, the trapezoidal rule and a stochastic process $\Delta\tilde{W}_{n,s}$, we propose a new explicit numerical scheme to solve the decoupled FBSDE (1.1) in the next section. Then in Section 3, after obtaining a useful inequality for the error estimate, the second-order convergence of our scheme is proved under some reasonable conditions on the coefficients of (1.1). We undertake some numerical calculations to demonstrate our theoretical results in Section 4, and give our conclusions in Section 5.

We first introduce some relevant notation as follows.

1. $|\cdot|$ denotes the standard Euclidean norm in the Euclidean space $\mathbb{R}$, $\mathbb{R}^q$ and $\mathbb{R}^{q\times d}$.
2. $L^2 = L^2_{\mathscr{F}}(0,T;\mathbb{R}^d)$ denotes the set of all $\mathscr{F}_t$-adapted and mean-square-integrable processes valued in $\mathbb{R}^d$.
3. $\mathscr{F}_s^{t,x}(t \le s \le T)$ denotes a σ-field generated by the diffusion process $\{X_r, t \le r \le s, X_t = x\}$. When $s = T$, we use $\mathscr{F}^{t,x}$ to denote $\mathscr{F}_T^{t,x}$.
4. $\mathbb{E}_s^{t,x}[\eta]$ denotes the conditional mathematical expectation of the random variable η under the σ-field $\mathscr{F}_s^{t,x}$, i.e., $\mathbb{E}_s^{t,x}[\eta] = \mathbb{E}[\eta|\mathscr{F}_s^{t,x}]$. When $s = t$, we use $\mathbb{E}_t^x[\eta]$ to denote $\mathbb{E}[\eta|\mathscr{F}_t^{t,x}]$.
5. $C_b^{l,k,k}$ denotes the set of continuously differential functions $\phi : (t,x,y) \in [0,T] \times \mathbb{R}^q \times \mathbb{R} \to \mathbb{R}$ with uniformly bounded partial derivatives $\partial_t^{l_1}\phi$ and $\partial_x^{k_1}\partial_y^{k_2}\phi$ for any positive integers $l_1 \le l$ and $k_1 + k_2 \le k$.
6. $C_b^{k_3,k_4}$ denotes the set of functions $\phi : (t,x) \in [0,T] \times \mathbb{R}^q \to \mathbb{R}$ with uniformly bounded partial derivatives up to k_3 with respect to t, and up to k_4 with respect to x.

Throughout, C stands for a generic positive constant depending only on T, the given data b, σ, f, ξ and the regularity of time partition, although its value may differ from place to place.

2. Explicit Numerical Scheme for the Decoupled FBSDE

For the time interval $[0,T]$, we consider the time partition

$$0 = t_0 < \cdots < t_N = T$$

with $\Delta t_n = t_{n+1} - t_n$ and $\Delta t = \max_{0\le n\le N-1} \Delta t_n$; and in order to establish error estimates, we assume that it satisfies the regularity constraint $\frac{\max_{0\le n\le N-1}\Delta t_n}{\min_{0\le n\le N-1}\Delta t_n} \le c_0$ where c_0 is a positive constant. We introduce a stochastic process $\Delta\tilde{W}_{n,s}$ defined by

$$\Delta\tilde{W}_{n,s} = 2\Delta W_{n,s} - \frac{3}{\Delta t_n}\int_{t_n}^{s}(r-t_n)dW_r, \text{ for } s \ge t_n\,, \tag{2.1}$$

where $\Delta W_{n,s} = W_s - W_{t_n}$. It is then assumed that $\Delta\tilde{W}_{n,s} = (\Delta\tilde{W}^1_{n,s},\cdots,\Delta\tilde{W}^d_{n,s})^*$ where $(\cdot)^*$ represents the transpose of $(\cdot)$, defining a d-dimensional Gaussian process with the following properties:

1. $\mathbb{E}^x_{t_n}[\Delta\tilde{W}_{n,s}] = 0$;
2. $\mathbb{E}^x_{t_n}[\Delta\tilde{W}^i_{n,s}\Delta\tilde{W}^j_{n,s}] = 0$ for $i \ne j$;
3. $\mathbb{E}^x_{t_n}[\Delta\tilde{W}^i_{n,s}\Delta W^i_{n,s}] = 2(s-t_n) - \frac{3(s-t_n)^2}{2\Delta t_n}$, and $\mathbb{E}^x_{t_n}[\Delta\tilde{W}^i_{n,t_{n+1}}\Delta W^i_{n,t_{n+1}}] = \frac{\Delta t_n}{2}$;
4. $\mathbb{E}^x_{t_n}[(\Delta\tilde{W}^i_{n,s}))^2] = 4(s-t_n) - \frac{6(s-t_n)^2}{\Delta t_n} + \frac{3(s-t_n)^3}{(\Delta t_n)^2}$, and $\mathbb{E}^x_{t_n}[(\Delta\tilde{W}^i_{n,t_{n+1}}))^2] = \Delta t_n$.

2.1. Reference equations

Let $(X^{t,x}_s, Y^{t,x}_s, Z^{t,x}_s)$ be the solution of (1.1) starting from time t with $X_t = x$ — i.e. $(X^{t,x}_s, Y^{t,x}_s, Z^{t,x}_s)$ satisfies

$$\begin{cases} X^{t,x}_s = x + \int_t^s b(r,X^{t,x}_r)dr + \int_t^s \sigma(r,X^{t,x}_r)dW_r\,, & s\in[t,T]\,, \\ Y^{t,x}_s = \xi + \int_s^T f(r,X^{t,x}_r,Y^{t,x}_r,Z^{t,x}_r)dr - \int_s^T Z^{t,x}_r dW_r\,, & s\in[t,T]\,. \end{cases} \tag{2.2}$$

Given the triple $(X^{t_n,x}_t, Y^{t_n,x}_t, Z^{t_n,x}_t)$ defined by (2.2) for $t\in[t_n,T]$, we therefore have

$$X^{t_n,x}_{t_{n+1}} = x + \int_{t_n}^{t_{n+1}} b(s,X^{t_n,x}_s)ds + \int_{t_n}^{t_{n+1}} \sigma(s,X^{t_n,x}_s)dW_s\,, \tag{2.3}$$

$$Y^{t_n,x}_{t_n} = Y^{t_n,x}_{t_{n+1}} + \int_{t_n}^{t_{n+1}} f^{t_n,x}_s ds - \int_{t_n}^{t_{n+1}} Z^{t_n,x}_s dW_s\,, \tag{2.4}$$

where $f^{t_n,x}_s = f(s,X^{t_n,x}_s,Y^{t_n,x}_s,Z^{t_n,x}_s)$, for $n = 0,1,\cdots,N-1$.

For the decoupled FBSDE, (2.3) from the forward SDE can be solved independently by existing numerical methods. In this article, we assume that $\{X_{t_{n+1}}^{t_n,x}\}$ can be represented via

$$X_{t_{n+1}}^{t_n,x} = x + \phi(t_n, x, \Delta t_n, \Delta W_{n,t_{n+1}}, \xi^{n+1}) + R_x^n \,, \tag{2.5}$$

where ϕ is some function and ξ^{n+1} a known vector Gaussian process, and R_x^n is the truncation error. Indeed, most commonly used numerical methods for SDE correspond to omitting the error term R_x^n from (2.5) — e.g. the Euler scheme, the Milstein scheme, and Itô-Taylor schemes [6]. Let us therefore consider (2.4) obtained from the BSDE, where on taking the conditional mathematical expectation $\mathbb{E}_{t_n}^x[\cdot]$ on both sides we obtain

$$Y_{t_n}^{t_n,x} = \mathbb{E}_{t_n}^x\left[Y_{t_{n+1}}^{t_n,x}\right] + \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x\left[f_s^{t_n,x}\right] ds \,. \tag{2.6}$$

On multiplying through by $\Delta\tilde{W}_{n,t_{n+1}}^*$ and taking the conditional mathematical expectation $\mathbb{E}_{t_n}^x[\cdot]$ on both sides of the consequent derived equation, we have

$$\begin{aligned} 0 =& \mathbb{E}_{t_n}^x\left[Y_{t_{n+1}}^{t_n,x}\Delta\tilde{W}_{n,t_{n+1}}^*\right] + \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x\left[f_s^{t_n,x}\Delta\tilde{W}_{n,t_{n+1}}^*\right] ds \\ &- \mathbb{E}_{t_n}^x\left[\int_{t_n}^{t_{n+1}} Z_s^{t_n,x} dW_s \cdot \Delta\tilde{W}_{n,t_{n+1}}^*\right] . \end{aligned} \tag{2.7}$$

Under the filtration $\mathscr{F}_{t_n}$, it is notable that the integrands in (2.6) and (2.7) are deterministic functions of s. Thus some numerical integration methods can be used to approximate the integrals in (2.6) and (2.7) accurately. From the trapezoidal rule, we obtain the identity

$$\int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x\left[f_s^{t_n,x}\right] ds = \frac{1}{2}\Delta t_n f_{t_n}^{t_n,x} + \frac{1}{2}\Delta t_n \mathbb{E}_{t_n}^x\left[f_{t_{n+1}}^{t_n,x}\right] + R_{yc}^n \,, \tag{2.8}$$

with the truncation error

$$R_{yc}^n = \int_{t_n}^{t_{n+1}} \left(\mathbb{E}_{t_n}^x\left[f_s^{t_n,x}\right] - \frac{1}{2} f_{t_n}^{t_n,x} - \frac{1}{2}\mathbb{E}_{t_n}^x\left[f_{t_{n+1}}^{t_n,x}\right]\right) ds \,.$$

Substituting (2.8) into (2.6), we deduce that

$$Y_{t_n}^{t_n,x} = \mathbb{E}_{t_n}^x\left[Y_{t_{n+1}}^{t_n,x}\right] + \frac{1}{2}\Delta t_n f_{t_n}^{t_n,x} + \frac{1}{2}\Delta t_n \mathbb{E}_{t_n}^x\left[f_{t_{n+1}}^{t_n,x}\right] + R_{yc}^n \,. \tag{2.9}$$

In proposing an explicit numerical scheme to solve for $\{Y_{t_n}^{t_n,x}\}$, we approximate the $Y_{t_n}^{t_n,x}$ in $f_{t_n}^{t_n,x} = f(t_n, X^n, Y_{t_n}^{t_n,x}, Z_{t_n}^{t_n,x})$ by the right rectangle formula

$$Y_{t_n}^{t_n,x} = \mathbb{E}_{t_n}^x\left[Y_{t_{n+1}}^{t_n,x}\right] + \Delta t_n \mathbb{E}_{t_n}^x\left[f_{t_{n+1}}^{t_n,x}\right] + R_{yr}^n \,, \tag{2.10}$$

where $R_{yr}^n = \int_{t_n}^{t_{n+1}} (\mathbb{E}_{t_n}^x[f_s^{t_n,x}] - \mathbb{E}_{t_n}^x[f_{t_{n+1}}^{t_n,x}])ds$. Thus we obtain the following reference equation to solve for $\{Y_{t_n}^{t_n,x}\}$ — viz.

$$Y_{t_n}^{t_n,x} = \mathbb{E}_{t_n}^x\left[Y_{t_{n+1}}^{t_n,x}\right] + \frac{1}{2}\Delta t_n \bar{f}_{t_n}^{t_n,x} + \frac{1}{2}\Delta t_n \mathbb{E}_{t_n}^x\left[f_{t_{n+1}}^{t_n,x}\right] + R_y^n, \tag{2.11}$$

where we have $\bar{f}_{t_n}^{t_n,x} = f(t_n, X_{t_n}^{t_n,x}, \mathbb{E}_{t_n}^x[Y_{t_{n+1}}^{t_n,x}] + \Delta t_n \mathbb{E}_{t_n}^x[f_{t_{n+1}}^{t_n,x}], Z_{t_n}^{t_n,x})$ and $R_y^n = R_{yc}^n + R_{yp}^n$ with $R_{yp}^n = \Delta t_n (f_{t_n}^{t_n,x} - \bar{f}_{t_n}^{t_n,x})/2$. Then from (2.7) and the definition of $\Delta\tilde{W}_{n,t_{n+1}}$, we introduce a reference equation to solve for $\{Z_{t_n}^{t_n,x}\}$ — viz.

$$\frac{1}{2}\Delta t_n Z_{t_n}^{t_n,x} = \mathbb{E}_{t_n}^x\left[Y_{t_{n+1}}^{t_n,x}\Delta\tilde{W}_{n,t_{n+1}}^*\right] + \Delta t_n \mathbb{E}_{t_n}^x\left[f_{t_{n+1}}^{t_n,x}\Delta\tilde{W}_{n,t_{n+1}}^*\right] + R_z^n, \tag{2.12}$$

where $R_z^n = R_{z1}^n + R_{z2}^n$ with $R_{z1}^n = \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x[f_s^{t_n,x}\Delta\tilde{W}_{n,t_{n+1}}^*]ds - \Delta t_n \mathbb{E}_{t_n}^x[f_{t_{n+1}}^{t_n,x}\Delta\tilde{W}_{n,t_{n+1}}^*]$ and $R_{z2}^n = \frac{1}{2}\Delta t_n Z_{t_n}^{t_n,x} - \mathbb{E}_{t_n}^x[\int_{t_n}^{t_{n+1}} Z_s^{t_n,x} dW_s \cdot \Delta\tilde{W}_{n,t_{n+1}}^*]$.

Remark 2.1. If the generator f, the terminal condition ξ, the drift coefficient b and the diffusion coefficient σ are sufficiently smooth, then $F(s) = \mathbb{E}_{t_n}^x[f_s^{t_n,x}]$ is a sufficiently smooth function of s. From the theory of numerical integration, the truncation terms R_{yc}^n and R_{yr}^n are $\mathcal{O}((\Delta t_n)^3)$ and $\mathcal{O}((\Delta t_n)^2)$, respectively — and further, $R_{yp}^n \sim \mathcal{O}((\Delta t_n)^3)$. Consequently, the truncation error term R_y^n in (2.11) is $\mathcal{O}((\Delta t_n)^3)$. In using the Gaussian process $\Delta\tilde{W}_{n,t}^*$, the truncation error term R_z^n in (2.12) is also $\mathcal{O}((\Delta t_n)^3)$. More detailed analysis of these terms is discussed in Section 3.

2.2. The explicit scheme

The triple (X^n, Y^n, Z^n) is used to denote the numerical approximation of the solution $(X_{t_n}, Y_{t_n}, Z_{t_n})$ to the decoupled FBSDE (1.1) at the time level $t = t_n (n = N, \cdots, 0)$. To simplify our presentation, we also use f^n to denote $f(t_n, X^n, Y^n, Z^n)$ when the context is clear. From the three reference equations (2.5), (2.11) and (2.12), we propose the following explicit scheme for solving the decoupled FBSDE (1.1):

Scheme 1. Given the random variables X_0, Y^N and Z^N, for $n = N-1, \cdots, 1, 0$ solve for the random variables Y^n and Z^n from

$$\frac{1}{2}\Delta t_n Z^n = \mathbb{E}_{t_n}^{X^n}\left[Y^{n+1}\Delta\tilde{W}_{n,t_{n+1}}^*\right] + \Delta t_n \mathbb{E}_{t_n}^{X^n}\left[f^{n+1}\Delta\tilde{W}_{n,t_{n+1}}^*\right], \tag{2.13}$$

$$\begin{cases} \bar{Y}^n = \mathbb{E}_{t_n}^{X^n}\left[Y^{n+1}\right] + \Delta t_n \mathbb{E}_{t_n}^{X^n}\left[f^{n+1}\right], \\ Y^n = \mathbb{E}_{t_n}^{X^n}\left[Y^{n+1}\right] + \frac{1}{2}\Delta t_n \bar{f}^n + \frac{1}{2}\Delta t_n \mathbb{E}_{t_n}^{X^n}\left[f^{n+1}\right], \end{cases} \tag{2.14}$$

where $\bar{f}^n = f(t_n, X^n, \bar{Y}^n, Z^n)$, $\Delta\tilde{W}_{n,t_{n+1}}$ is defined by (2.1) with $s = t_{n+1}$, and the X^{n+1} used in (2.13) and (2.14) is rendered by

$$X^{n+1} = X^n + \phi\left(t_n, X^n, \Delta t_n, \Delta W_{n,t_{n+1}}, \xi^{n+1}\right). \tag{2.15}$$

In Scheme 1, we first predict the value $\bar{Y}^n$ of Y^n, and then use this value to solve for Y^n from (2.14). The computing cost is notably less than the implicit numerical schemes [15,17,18]. Its accuracy of course depends not only on the accuracy of the scheme (2.13) and (2.14) for the BSDE in (2.2), but also on the accuracy of the scheme (2.15) for the SDE in (2.2). As discussed in Section 3, it turns out that Scheme 1 is second-order when weak second-order methods are applied to solve the forward SDE.

The random variables $\Delta W_{n,t_{n+1}}$, ξ^{n+1} and $\Delta\tilde{W}_{n,t_{n+1}}$ are correlated, and they can be orthogonalised in any simulation via the Gram-Schmidt process. For example, when ϕ does not depend on ξ^n, $\Delta W_{n,t_{n+1}}$ and $\Delta\tilde{W}_{n,t_{n+1}}$ can be simulated by

$$\Delta W_{n,t_{n+1}} = \sqrt{\Delta t_n} N(0,1)\,, \qquad \Delta\tilde{W}_{n,t_{n+1}} = \frac{\sqrt{\Delta t_n}}{2}\left(N(0,1) - \sqrt{3}\tilde{N}(0,1)\right)\,, \tag{2.16}$$

where $N(0,1)$ and $\tilde{N}(0,1)$ are two independent standard normal random variables. The conditional mathematical expectations $\mathbb{E}_{t_n}^{X^n}[Y^{n+1}]$, $\mathbb{E}_{t_n}^{X^n}[f^{n+1}]$, $\mathbb{E}_{t_n}^{X^n}[Y^{n+1}\Delta\tilde{W}^*_{n,t_{n+1}}]$, and $\mathbb{E}_{t_n}^{X^n}[f^{n+1}\Delta\tilde{W}^*_{n,t_{n+1}}]$ should be calculated by some numerical procedure, such as a Monte-Carlo method or Gauss-type quadrature — cf. Refs. [15,17] for more details.

3. Error Estimates

Let $\tilde{Y}_{t_{n+1}}^{t_n,X^n}$ and $\tilde{Z}_{t_{n+1}}^{t_n,X^n}$ denote the respective values of $Y_{t_{n+1}}^{t_n,X^n}$ and $Z_{t_{n+1}}^{t_n,X^n}$ at X^{n+1}. For convenience, we introduce the following notation:

$$\begin{aligned}
&\tilde{f}_{t_{n+1}}^{t_n,X^n} = f\left(t_{n+1}, X^{n+1}, \tilde{Y}_{t_{n+1}}^{t_n,X^n}, \tilde{Z}_{t_{n+1}}^{t_n,X^n}\right)\,,\\
&\hat{f}_{t_n}^{t_n,X^n} = f\left(t_n, X^n, \mathbb{E}_{t_n}^{X^n}\left[\tilde{Y}_{t_{n+1}}^{t_n,X^n}\right] + \Delta t_n \mathbb{E}_{t_n}^{X^n}\left[\tilde{f}_{t_{n+1}}^{t_n,X^n}\right], \tilde{Z}_{t_n}^{t_n,X^n}\right)\,,\\
&e_y^n = Y_{t_n}^{t_n,X^n} - Y^n, \qquad e_z^n = Z_{t_n}^{t_n,X^n} - Z^n\,,\\
&e_f^{n+1} = \tilde{f}_{t_{n+1}}^{t_n,X^n} - f^{n+1}, \qquad e_{\hat{f}}^n = \hat{f}_{t_n}^{t_n,X^n} - \bar{f}^n\,.
\end{aligned}$$

From the definitions of $\tilde{Y}_{t_{n+1}}^{t_n,X^n}$ and $\tilde{Z}_{t_{n+1}}^{t_n,X^n}$, we then have that

$$\tilde{Y}_{t_{n+1}}^{t_n,X^n} - Y^{n+1} = e_y^{n+1}, \qquad \tilde{Z}_{t_{n+1}}^{t_n,X^n} - Z^{n+1} = e_z^{n+1}\,.$$

3.1. Stability error estimate

We now introduce the following notation:

$$\begin{aligned}
&R_{y_1}^n = \mathbb{E}_{t_n}^{X^n}\left[Y_{t_{n+1}}^{t_n,X^n} - \tilde{Y}_{t_{n+1}}^{t_n,X^n}\right]\,, \qquad R_{y_2}^n = \mathbb{E}_{t_n}^{X^n}\left[f_{t_{n+1}}^{t_n,X^n} - \tilde{f}_{t_{n+1}}^{t_n,X^n}\right]\,,\\
&R_{y_3}^n = \mathbb{E}_{t_n}^{X^n}\left[\bar{f}_{t_n}^{t_n,X^n} - \hat{f}_{t_n}^{t_n,X^n}\right]\,, \qquad R_{z_1}^n = \mathbb{E}_{t_n}^{X^n}\left[(Y_{t_{n+1}}^{t_n,X^n} - \tilde{Y}_{t_{n+1}}^{t_n,X^n})\Delta\tilde{W}^*_{n,t_{n+1}}\right]\,,\\
&R_{z_2}^n = \mathbb{E}_{t_n}^{X^n}\left[(f_{t_{n+1}}^{t_n,X^n} - \tilde{f}_{t_{n+1}}^{t_n,X^n})\Delta\tilde{W}^*_{n,t_{n+1}}\right]\,.
\end{aligned} \tag{3.1}$$

Our stability error estimate for Scheme 1 is summarised in the following theorem.

Theorem 3.1. *Let (X_t, Y_t, Z_t), $t \in [0, T]$ satisfy the decoupled FBSDE* (1.1) *and (X^n, Y^n, Z^n) for $n = 0, 1, \cdots, N-1$ denote its numerical solution by Scheme* 1*, for a regular time partition with constraint constant c_0. Assume that the generator $f(t,x,y,z)$ is Lipschitz continuous with respect to x, y and z with Lipschitz constant L. Then for sufficiently small time step Δt,*

$$
\begin{aligned}
&\mathbb{E}\Big[|e_y^n|^2\Big] + \Delta t \sum_{i=n}^{N-1} (1 + C\Delta t)^{i-n} \mathbb{E}\Big[|e_z^i|^2\Big] \\
\leq & C\left(\mathbb{E}[|e_y^N|^2] + \Delta t \mathbb{E}[|e_z^N|^2]\right) \\
&+ \sum_{i=n}^{N-1} (1 + C\Delta t)^{i-n} \frac{C\mathbb{E}\Big[|R_{y_1}^i|^2 + (\Delta t)^2(|R_{y_2}^i|^2 + |R_{y_3}^i|^2) + |R_y^i|^2\Big]}{\Delta t} \\
&+ \sum_{i=n}^{N-1} (1 + C\Delta t)^{i-n} C\Delta t \mathbb{E}\left[\left(\frac{1}{\Delta t_n}\right)^2 |R_{z_1}^i|^2 + |R_{z_2}^i|^2 + \left(\frac{1}{\Delta t_n}\right)^2 |R_z^i|^2\right]
\end{aligned} \tag{3.2}
$$

for $n = N-1, \cdots, 1, 0$. Here C is a positive constant depending on T, c_0 and L; and the terms R_y^i, R_z^i, $R_{y_1}^i$, $R_{y_2}^i$, $R_{y_3}^i$, $R_{z_1}^i$ and $R_{z_2}^i$ are defined in (2.11), (2.12) and (3.1)*, respectively.*

Proof. We prove this theorem in three steps.

Step 1. The estimate of e_y^n.

For each integer n $(0 \leq n \leq N-1)$, subtracting (2.14) from (2.11) gives

$$
\begin{aligned}
e_y^n =& \mathbb{E}_{t_n}^{X^n}\Big[Y_{t_{n+1}}^{t_n,X^n} - Y^{n+1}\Big] + \frac{1}{2}\Delta t_n \left(\bar{f}_{t_n}^{t_n,X^n} - \bar{f}^n\right) + \frac{1}{2}\Delta t_n \mathbb{E}_{t_n}^{X^n}\Big[f_{t_{n+1}}^{t_n,X^n} - f^{n+1}\Big] + R_y^n \\
=& \mathbb{E}_{t_n}^{X^n}\Big[Y_{t_{n+1}}^{t_n,X^n} - \tilde{Y}_{t_{n+1}}^{t_n,X^n} + \tilde{Y}_{t_{n+1}}^{t_n,X^n} - Y^{n+1}\Big] + \frac{\Delta t_n}{2}\left(\bar{f}_{t_n}^{t_n,X^n} - \hat{f}_{t_n}^{t_n,X^n} + \hat{f}_{t_n}^{t_n,X^n} - \bar{f}^n\right) \\
&+ \frac{\Delta t_n}{2}\mathbb{E}_{t_n}^{X^n}\Big[f_{t_{n+1}}^{t_n,X^n} - \tilde{f}_{t_{n+1}}^{t_n,X^n} + \tilde{f}_{t_{n+1}}^{t_n,X^n} - f^{n+1}\Big] + R_y^n \\
=& \mathbb{E}_{t_n}^{X^n}\Big[e_y^{n+1}\Big] + \frac{\Delta t_n}{2} e_{\hat{f}}^n + \frac{\Delta t_n}{2}\mathbb{E}_{t_n}^{X^n}\Big[e_f^{n+1}\Big] + R_{y_1}^n + \frac{\Delta t_n}{2}\left(R_{y_2}^n + R_{y_3}^n\right) + R_y^n\,.
\end{aligned} \tag{3.3}
$$

Then from the properties $|e_{\hat{f}}^n| \leq L((1+L\Delta t_n)(\mathbb{E}_{t_n}^{X^n}[|e_y^{n+1}|] + \mathbb{E}_{t_n}^{X^n}[|e_z^{n+1}|]) + |e_z^n|)$ and $|\mathbb{E}_{t_n}^{X^n}[e_f^{n+1}]| \leq L(\mathbb{E}_{t_n}^{X^n}[|e_y^{n+1}|] + \mathbb{E}_{t_n}^{X^n}[|e_z^{n+1}|])$, it follows that

$$
\begin{aligned}
|e_y^n| \leq& \left|\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}]\right| + \frac{L\Delta t_n}{2}|e_z^n| + L\Delta t_n\left(1 + \frac{L\Delta t_n}{2}\right)\mathbb{E}_{t_n}^{X^n}\Big[|e_y^{n+1}| + |e_z^{n+1}|\Big] \\
&+ |R_{y_1}^n| + \frac{\Delta t_n}{2}\left(|R_{y_2}^n| + |R_{y_3}^n|\right) + |R_y^n|\,.
\end{aligned} \tag{3.4}
$$

On noting the two inequalities

$$
(a+b)^2 \leq (1 + \gamma\Delta t)a^2 + \left(1 + \frac{1}{\gamma\Delta t}\right)b^2, \qquad \left(\sum_{n=1}^m a_n\right)^2 \leq m\sum_{n=1}^m a_n^2\,, \tag{3.5}
$$

we then obtain the estimate for any positive real number γ and positive integer m:

$$
\begin{aligned}
|e_y^n|^2 \le & (1+\gamma\Delta t)\left|\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}]\right|^2 + \left(1+\frac{1}{\gamma\Delta t}\right)\left\{\frac{L\Delta t_n}{2}|e_z^n|\right. \\
& \left. + L\Delta t_n\left(1+\frac{L\Delta t_n}{2}\right)\mathbb{E}_{t_n}^{X^n}\left[|e_y^{n+1}|+|e_z^{n+1}|\right] + |R_{y_1}^n| + \frac{\Delta t_n}{2}\left(|R_{y_2}^n|+|R_{y_3}^n|\right)+|R_y^n|\right\}^2 \\
\le & (1+\gamma\Delta t)\left|\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}]\right|^2 + \left\{\frac{5L^2(\Delta t)^2}{4}|e_z^n|^2\right. \\
& + 10L^2(\Delta t)^2\left(1+\frac{L\Delta t}{2}\right)^2\mathbb{E}_{t_n}^{X^n}\left[|e_y^{n+1}|^2+|e_z^{n+1}|^2\right] \\
& \left. +5|R_{y_1}^n|^2 + \frac{5\Delta t^2}{2}\left(|R_{y_2}^n|^2+|R_{y_3}^n|^2\right)+5|R_y^n|^2\right\} \\
& + \frac{5}{\gamma}\left\{\frac{L^2\Delta t}{4}|e_z^n|^2 + 2L^2\Delta t\left(1+\frac{L\Delta t}{2}\right)^2\mathbb{E}_{t_n}^{X^n}\left[|e_y^{n+1}|^2+|e_z^{n+1}|^2\right]\right\} \\
& + \frac{1}{\gamma\Delta t}\left\{5|R_{y_1}^n|^2 + \frac{5(\Delta t)^2}{2}\left(|R_{y_2}^n|^2+|R_{y_3}^n|^2\right)+5|R_y^n|^2\right\}.
\end{aligned}
\tag{3.6}
$$

Step 2. The estimate of e_z^n.

From (2.12) and (2.13),

$$
\begin{aligned}
\frac{\Delta t_n}{2}e_z^n = & \mathbb{E}_{t_n}^{X^n}\left[(Y_{t_{n+1}}^{t_n,X^n}-Y^{n+1})\Delta\tilde{W}_{n,t_{n+1}}^*\right] \\
& + \Delta t_n\mathbb{E}_{t_n}^{X^n}\left[(f_{t_{n+1}}^{t_n,X^n}-f^{n+1})\Delta\tilde{W}_{n,t_{n+1}}^*\right] + R_z^n.
\end{aligned}
\tag{3.7}
$$

As for (3.3), we get the identifies

$$
\begin{aligned}
\mathbb{E}_{t_n}^{X^n}\left[(Y_{t_{n+1}}^{t_n,X^n}-Y^{n+1})\Delta\tilde{W}_{n,t_{n+1}}^*\right] &= R_{z_1}^n + \mathbb{E}_{t_n}^{X^n}\left[e_y^{n+1}\Delta\tilde{W}_{n,t_{n+1}}^*\right], \\
\mathbb{E}_{t_n}^{X^n}\left[(f_{t_{n+1}}^{t_n,X^n}-f^{n+1})\Delta\tilde{W}_{n,t_{n+1}}^*\right] &= R_{z_2}^n + \mathbb{E}_{t_n}^{X^n}\left[e_f^{n+1}\Delta\tilde{W}_{n,t_{n+1}}^*\right],
\end{aligned}
$$

and inserting them into (3.7) gives

$$
e_z^n = \frac{2}{\Delta t_n}\mathbb{E}_{t_n}^{X^n}\left[e_y^{n+1}\Delta\tilde{W}_{n,t_{n+1}}^*\right] + 2\mathbb{E}_{t_n}^{X^n}\left[e_f^{n+1}\Delta\tilde{W}_{n,t_{n+1}}^*\right] + \frac{2}{\Delta t_n}R_{z_1}^n + 2R_{z_2}^n + \frac{2}{\Delta t_n}R_z^n.
$$

Consequently we have the estimate

$$
\begin{aligned}
|e_z^n| \le & \frac{2}{\Delta t_n}\left|\mathbb{E}_{t_n}^{X^n}\left[e_y^{n+1}\Delta\tilde{W}_{n,t_{n+1}}^*\right]\right| + 2\left|\mathbb{E}_{t_n}^{X^n}\left[e_f^{n+1}\Delta\tilde{W}_{n,t_{n+1}}^*\right]\right| \\
& + \frac{2}{\Delta t_n}|R_{z_1}^n| + 2|R_{z_2}^n| + \frac{2}{\Delta t_n}|R_z^n|.
\end{aligned}
\tag{3.8}
$$

From Hölder's inequality and (3.5) with $\gamma\Delta t$ replaced by any positive real number ε, it follows that

$$\begin{aligned}|e_z^n|^2 \le&(1+\varepsilon)\left(\frac{2}{\Delta t_n}\right)^2\left|\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}\Delta\tilde{W}_{n,t_{n+1}}^*]\right|^2+\left(1+\frac{1}{\varepsilon}\right)\Big\{2\left|\mathbb{E}_{t_n}^{X^n}[e_f^{n+1}\Delta\tilde{W}_{n,t_{n+1}}^*]\right|\\&+\frac{2}{\Delta t_n}|R_{z_1}^n|+2|R_{z_2}^n|+\frac{2}{\Delta t_n}|R_z^n|\Big\}^2\\ \le&(1+\varepsilon)\left(\frac{2}{\Delta t_n}\right)^2\left|\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}\Delta\tilde{W}_{n,t_{n+1}}^*]\right|^2+16\left(1+\frac{1}{\varepsilon}\right)\Big\{\mathbb{E}_{t_n}^{X^n}\left[|e_f^{n+1}|^2\right]\mathbb{E}_{t_n}^{X^n}\left[|\Delta\tilde{W}_{n,t_{n+1}}^*|^2\right]\\&+\frac{1}{(\Delta t_n)^2}|R_{z_1}^n|^2+|R_{z_2}^n|^2+\frac{1}{(\Delta t_n)^2}|R_z^n|^2\Big\}\ .\end{aligned}\tag{3.9}$$

For the three conditional expectations in (3.9), we have the estimates

$$\begin{aligned}&\mathbb{E}_{t_n}^{X^n}\left[|\Delta\tilde{W}_{n,t_{n+1}}^*|^2\right]=d\Delta t_n\ ,\\&\mathbb{E}_{t_n}^{X^n}\left[|e_f^{n+1}|^2\right]\le\mathbb{E}_{t_n}^{X^n}\left[|L(|e_y^{n+1}|+|e_z^{n+1}|)|^2\right]\le 2L^2\mathbb{E}_{t_n}^{X^n}\left[|e_y^{n+1}|^2+|e_z^{n+1}|^2\right]\ ,\\&\mathbb{E}_{t_n}^{X^n}\left|[e_y^{n+1}\Delta\tilde{W}_{n,t_{n+1}}^*]\right|^2=\left|\mathbb{E}_{t_n}^{X^n}\left[(e_y^{n+1}-\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}])\Delta\tilde{W}_{n,t_{n+1}}^*\right]\right|^2\\&\quad\le\mathbb{E}_{t_n}^{X^n}\left[|\Delta\tilde{W}_{n,t_{n+1}}^*|^2\right]\mathbb{E}_{t_n}^{X^n}\left[\left(e_y^{n+1}-\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}]\right)^2\right]=d\Delta t_n\left(\mathbb{E}_{t_n}^{X^n}\left[|e_y^{n+1}|^2\right]-\left|\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}]\right|^2\right).\end{aligned}$$

Substituting these three estimates into (3.9), we obtain

$$\begin{aligned}|e_z^n|^2\le&(1+\varepsilon)\frac{4d}{\Delta t_n}\left(\mathbb{E}_{t_n}^{X^n}\left[|e_y^{n+1}|^2\right]-\left|\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}]\right|^2\right)\\&+32\left(1+\frac{1}{\varepsilon}\right)L^2d\Delta t_n\mathbb{E}_{t_n}^{X^n}\left[|e_y^{n+1}|^2+|e_z^{n+1}|^2\right]\\&+16\left(1+\frac{1}{\varepsilon}\right)\left(\frac{1}{(\Delta t_n)^2}|R_{z_1}|^2+|R_{z_2}^n|^2+\frac{1}{(\Delta t_n)^2}|R_z^n|^2\right)\ .\end{aligned}\tag{3.10}$$

On dividing both sides of (3.10) by $(1+\varepsilon)4d/\Delta t$, we conclude that

$$\begin{aligned}\frac{\Delta t}{4d(1+\varepsilon)}|e_z^n|^2\le&c_0\left(\mathbb{E}_{t_n}^{X^n}\left[|e_y^{n+1}|^2\right]-\left|\mathbb{E}_{t_n}^{X^n}[e_y^{n+1}]\right|^2\right)+\frac{8L^2}{\varepsilon}\Delta t_n\Delta t\,\mathbb{E}_{t_n}^{X^n}\left[|e_y^{n+1}|^2+|e_z^{n+1}|^2\right]\\&+\frac{4\Delta t}{d\varepsilon}\left(\frac{1}{(\Delta t_n)^2}|R_{z_1}^n|^2+|R_{z_2}^n|^2+\frac{1}{(\Delta t_n)^2}|R_z^n|^2\right)\ .\end{aligned}\tag{3.11}$$

Step 3. The proof of (3.2).

Multiplying (3.6) by c_0 and then adding the result to (3.11), we obtain

$$
\begin{aligned}
& c_0|e_y^n|^2+\frac{\Delta t}{4(1+\varepsilon)}|e_z^n|^2 \\
\leq & c_0\left(1+\left(\gamma+10L^2\left(1+\frac{L\Delta t}{2}\right)^2\left(\frac{1}{\gamma}+\Delta t\right)+\frac{8L^2\Delta t}{c_0\varepsilon}\right)\Delta t\right)\mathbb{E}_{t_n}^{X^n}\left[|e_y^{n+1}|^2\right] \\
& +\left(10c_0\left(1+\frac{L\Delta t}{2}\right)^2\left(\frac{1}{\gamma}+\Delta t\right)+\frac{8\Delta t}{d\varepsilon}\right)L^2\Delta t\mathbb{E}_{t_n}^{X^n}\left[|e_z^{n+1}|^2\right] \\
& +\frac{5c_0L^2}{4}\left(\frac{1}{\gamma}+\Delta t)\right)\Delta t|e_z^n|^2+5c_0\left(1+\frac{1}{\gamma\Delta t}\right)\left\{|R_{y_1}^n|^2+\frac{1}{2}\Delta t^2\left(|R_{y_2}^n|^2+|R_{y_3}^n|^2\right)+|R_y^n|^2\right\} \\
& +\frac{4\Delta t}{d\varepsilon}\left\{\left(\frac{1}{\Delta t_n}\right)^2|R_{z_1}^n|^2+|R_{z_2}^n|^2+\left(\frac{1}{\Delta t_n}\right)^2|R_z^n|^2\right\},
\end{aligned}
$$

which can be simplified to

$$
\begin{aligned}
& c_0\mathbb{E}\left[|e_y^n|^2\right]+C_3\Delta t\mathbb{E}\left[|e_z^n|^2\right] \\
\leq & c_0[1+C_2\Delta t]\mathbb{E}\left[|e_y^{n+1}|^2\right]+C_4\Delta t\mathbb{E}\left[|e_z^{n+1}|^2\right] \\
& +\frac{C_5}{\Delta t}\mathbb{E}\left[|R_{y_1}^n|^2+\frac{1}{2}\Delta t^2\left(|R_{y_2}^n|^2+|R_{y_3}^n|^2\right)+|R_y^n|^2\right] \\
& +\frac{C_6\Delta t}{\varepsilon}\mathbb{E}\left[\left(\frac{1}{\Delta t_n}\right)^2|R_{z_1}^n|^2+|R_{z_2}^n|^2+\left(\frac{1}{\Delta t_n}\right)^2|R_z^n|^2\right],
\end{aligned}
$$

where

$$
\begin{aligned}
C_2 &= \left(\gamma+10L^2\left(1+\frac{L\Delta t}{2}\right)^2\left(\frac{1}{\gamma}+\Delta t\right)+8L^2\Delta t/(c_0\varepsilon)\right)\Delta t, \\
C_3 &= \frac{1}{4(1+\varepsilon)}-\frac{5c_0L^2}{4}\left(\frac{1}{\gamma}+\Delta t\right), \\
C_4 &= \left(10c_0\left(1+\frac{L\Delta t}{2}\right)^2\left(\frac{1}{\gamma}+\Delta t\right)+\frac{8\Delta t}{d\varepsilon}\right)L^2, \\
C_5 &= 5c_0\frac{1+\gamma\Delta t}{\gamma}, \qquad C_6=\frac{4}{d}.
\end{aligned}
$$

Now choose $\varepsilon=1$, γ large enough and Δt_0 sufficiently small such that if $0<\Delta t\leq\Delta t_0$ there exist two positive constants C and C^* depending on c_0 and L satisfying $C_2\leq C$, $C_5\leq C$, and $C_3-C_4>C^*>0$. Then for $0<\Delta t\leq\Delta t_0$, we obtain

$$
\begin{aligned}
& c_0\mathbb{E}\left[|e_y^n|^2\right]+C_3\Delta t\mathbb{E}\left[|e_z^n|^2\right] \\
\leq & c_0(1+C\Delta t)\mathbb{E}\left[|e_y^{n+1}|^2\right]+C_4\Delta t\mathbb{E}\left[|e_z^{n+1}|^2\right] \\
& +\frac{C\mathbb{E}\left[|R_{y_1}^n|^2+\frac{1}{2}(\Delta t)^2(|R_{y_2}^n|^2+|R_{y_3}^n|^2)+|R_y^n|^2\right]}{\Delta t} \\
& +C_6\Delta t\mathbb{E}\left[\left(\frac{1}{\Delta t_n}\right)^2|R_{z_1}^n|^2+|R_{z_2}^n|^2+\left(\frac{1}{\Delta t_n}\right)^2|R_z^n|^2\right].
\end{aligned}
\tag{3.12}
$$

On substituting e_y^i, $i = n+1, \cdots, N-1$ recursively, we consequently deduce that

$$
\begin{aligned}
&c_0 \mathbb{E}\Big[|e_y^n|^2\Big] + C_3 \Delta t \sum_{i=n}^{N-1} (1 + C\Delta t)^{i-n} \mathbb{E}\Big[|e_z^i|^2\Big] \\
\leq &(1 + C\Delta t)^{N-n} c_0 \mathbb{E}\Big[|e_y^N|^2\Big] + C_4 \Delta t \sum_{i=n+1}^{N} (1 + C\Delta t)^{i-n} \mathbb{E}\Big[|e_z^i|^2\Big] \\
&+ \sum_{i=n}^{N-1} (1 + C\Delta t)^{i-n} \frac{C\mathbb{E}\Big[|R_{y_1}^i|^2 + (\Delta t)^2(|R_{y_2}^n|^2 + |R_{y_2}^n|^2) + |R_y^i|^2\Big]}{\Delta t} \\
&+ \sum_{i=n}^{N-1} (1 + C\Delta t)^{i-n} C_6 \Delta t \mathbb{E}\left[\left(\frac{1}{\Delta t_n}\right)^2 |R_{z_1}^i|^2 + |R_{z_2}^i|^2 + \left(\frac{1}{\Delta t_n}\right)^2 |R_z^i|^2\right]
\end{aligned} \tag{3.13}
$$

— i.e.

$$
\begin{aligned}
&c_0 \mathbb{E}\Big[|e_y^n|^2\Big] + C^* \Delta t \sum_{i=n}^{N-1} (1 + C\Delta t)^{i-n} \mathbb{E}\Big[|e_z^i|^2\Big] \\
\leq &(1 + C\Delta t)^{N-n} c_0 \mathbb{E}\Big[|e_y^N|^2\Big] + C_4 \Delta t (1 + C\Delta t)^{N-n} \mathbb{E}\Big[|e_z^N|^2\Big] \\
&+ \sum_{i=n}^{N-1} (1 + C\Delta t)^{i-n} \frac{C\mathbb{E}\Big[|R_{y_1}^i|^2 + (\Delta t)^2(|R_{y_2}^i|^2 + |R_{y_3}^i|^2) + |R_y^i|^2\Big]}{\Delta t} \\
&+ \sum_{i=n}^{N-1} (1 + C\Delta t)^{i-n} C_6 \Delta t \mathbb{E}\left[\left(\frac{1}{\Delta t_n}\right)^2 |R_{z_1}^i|^2 + |R_{z_2}^i|^2 + \left(\frac{1}{\Delta t_n}\right)^2 |R_z^i|^2\right],
\end{aligned}
$$

which leads to the inequality (3.2). $\square$

Remark 3.1.

- The estimate (3.2) in Theorem 3.1 implies that Scheme 1 is stable.
- The terms R_y^n and R_z^n are the truncated errors from the numerical integration methods used in (2.11) and (2.12). The five terms $R_{y_1}^n$, $R_{y_2}^n$, $R_{y_3}^n$, $R_{z_1}^n$ and $R_{z_2}^n$ are determined by the numerical approximation (2.15) for solving the SDE in (1.1), which reflect the local errors (in weak sense) of the SDE scheme. Under certain regularity conditions on b, σ, f and ξ, the estimates of these terms can be obtained, then we can get the error estimates for Scheme 1 by Theorem 3.1.

3.2. Error estimates for the explicit scheme

Let us now consider the error estimate of Scheme 1 for the decoupled FBSDE (1.1) with $\xi = \varphi(X_T)$ under some regularity conditions on the functions b, σ, f and φ. We first derive estimates for the error terms R_y^n, R_z^n, $R_{y_1}^n$, $R_{y_2}^n$, $R_{y_3}^n$, $R_{z_1}^n$ and $R_{z_2}^n$ defined in Theorem 3.1, and then get the error estimate of Scheme 1 from Theorem 3.1.

To proceed, we introduce the following assumption:

Assumption 3.1. Assume X_{t_0} is $\mathscr{F}_{t_0}$-measurable with $\mathbb{E}[|X_{t_0}|^2] < \infty$. The drift coefficient b and the diffusion coefficient σ are jointly L^2-measurable in $(t,x) \in [t_0, T] \times \mathbb{R}^q$, uniformly Lipschitz continuous and of linear growth — i.e. there exist constants $L > 0$ and $K > 0$ such that

$$|b(t,x)|^2 \le K(1+|x|^2)\,, \qquad |\sigma(t,x)|^2 \le K(1+|x|^2)\,, \tag{3.14}$$

$$|b(t,x)-b(t,y)| \le L|x-y|\,, \qquad |\sigma(t,x)-\sigma(t,y)| \le L|x-y|\,, \tag{3.15}$$

for all $t,s \in [0,T]$ and $x,y \in \mathbb{R}^q$.

It is well-known that under Assumption 3.1 the SDE in (1.1) has a unique solution [12], and the following lemma shows some regularity results for the solution (Y_t, Z_t) of the decoupled FBSDE (1.1).

Lemma 3.1. (cf. Refs. [2,4,5,10,14]) *Let the functions b, σ, f and φ be uniformly Lipschitz continuous with respect to (x,y,z) and Hölder continuous with respect to t with parameter $1/2$. Assume $\varphi \in C_b^{2+\alpha}$ for some $\alpha \in (0,1)$, and that the matrix-valued function $a = \sigma\sigma^*$ is uniformly elliptic. Then the solution (Y_t, Z_t) of (1.1) can be represented as $Y_t = u(t,X_t)$ and $Z_t = \nabla_x u(t,X_t)\sigma(t,X_t)$, where $u(t,x)$ is the smooth solution of the PDE*

$$(\partial_t + \mathscr{L}_{t,x})u(t,x) + f\,(t,x,u(t,x),\nabla_x u(t,x)\sigma(t,x)) = 0$$

with the terminal condition $u(T,x) = \varphi(x)$, where $\mathscr{L}$ is the second order differential operator defined by

$$\mathscr{L}_{t,x} = \frac{1}{2}\sum_{i,j}[\sigma\sigma^*]_{ij}(t,x)\partial^2_{x_i x_j} + \sum_i b_i(t,x)\partial_{x_i}\,.$$

Furthermore, if $b,\sigma \in C_b^{1+k,2+2k}$, $f \in C_b^{1+k,2+2k,2+2k}$ and $\varphi \in C_b^{2+2k+\alpha}$ for some $\alpha \in (0,1)$ where $k = 0,1,2,\cdots$, then $u \in C_b^{1+k,2+2k}$ for $k = 0,1,2,\cdots$.

Lemma 3.1 states that under suitable conditions the solution Y_t and Z_t of (1.1) are functions of (t,X_t), and smooth functions if b, σ, f and φ are smooth.

Obviously, the accuracy of Scheme 1 depends on the accuracy of the numerical scheme for solving the forward SDE, and we make the following assumption on the scheme (2.15).

Assumption 3.2. The numerical solution X^n solved by (2.15) has the following stability property: for any positive integer r, there exists a constant $C \in (0,\infty)$ such that

$$\max_{0\le n\le N} \mathbb{E}\,[|X^n|^r] \le C\,(1+\mathbb{E}[|X_0|^r]) \tag{3.16}$$

— and the following approximation properties: there exist positive numbers r_1, r_2, β, γ such that for any $g \in C_b^{2\beta+2}$ and for $n = 0,1,\cdots,N-1$ we have

$$\left|\mathbb{E}_{t_n}^{X^n}\left[g(X_{t_{n+1}}^{t_n,x}) - g(X^{n+1})\right]\right| \le C_g(1+|X^n|^{2r_1})(\Delta t)^{\beta+1}\,, \tag{3.17}$$

$$\left|\mathbb{E}_{t_n}^{X^n}\left[(g(X_{t_{n+1}}^{t_n,x}) - g(X^{n+1}))\Delta\tilde{W}_{t_{n+1}}\right]\right| \le C_g(1+|X^n|^{2r_2})(\Delta t)^{\gamma+1}\,, \tag{3.18}$$

$$\left|\mathbb{E}\left[g(X_{t_n}) - g(X^n)\right]\right| \le C_g(\Delta t)^{\beta}\,, \tag{3.19}$$

where $C_g > 0$ is a constant that does not depend on Δt. The number $\beta + 1$ is called the local weak order of the approximation.

Remark 3.2. Many numerical schemes for forward SDE have the above properties — e.g. for the Euler and Milstein schemes, with $\beta = \gamma = 1$; and for the weak order-2.0 Itô-Taylor scheme [6], with $\beta = \gamma = 2$. Also, under Assumption 3.1, if $\mathbb{E}[|X_0|^{2m}] < \infty$ for some integer $m \geq 1$ then the solution of (2.3) has the estimate

$$\mathbb{E}_{t_n}^{X^n}(|X_s^{t_n,X^n}|^{2m}) \leq (1 + |X^n|^{2m})e^{C(s-t_n)} \tag{3.20}$$

for any $s \in [t_n, T]$, where C is a positive constant depending only on the constants K, L and m.

We now estimate the truncation errors R_y^n, R_z^n, $R_{y_1}^n$, $R_{y_2}^n$, $R_{y_3}^n$, $R_{z_1}^n$ and $R_{z_2}^n$ under certain regularity conditions on b, σ, f and φ. For clarity, we only consider the case $q = d = 1$. Let L^0 and L^1 be two differential operators defined by

$$L^1 = \sigma \partial_x , \qquad L^0 = \partial_t + b\partial_x + \frac{1}{2}\sigma^2 \partial_{xx}^2 .$$

We introduce the following two lemmas.

Lemma 3.2. *If $f(t,x,y,z) \in C_b^{2,4,4,4}$, $b(t,x), \sigma(t,x) \in C_b^{2,4}$, $\varphi \in C_b^{4+\alpha}$ for $\alpha \in (0,1)$ and $|b(t,x)|^2 \leq K(1+|x|^2)$, $|\sigma(t,x)|^2 \leq K(1+|x|^2)$, then for sufficiently small time step Δt we have*

$$\mathbb{E}\Big[|R_y^n|^2\Big] \leq C\left(1 + \mathbb{E}[|X^n|^8]\right)(\Delta t)^6 , \quad \textit{for any } 0 \leq n \leq N-1 , \tag{3.21}$$

where C is a positive constant depending only on T, K and the upper bounds of the derivatives of b, σ, f and φ.

Lemma 3.3. *Under the conditions of Lemma* 3.2*, for sufficiently small time step Δt and for any $0 \leq n \leq N-1$,*

$$\begin{aligned}
|R_{z1}^n| &= \left| \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{X^n}[f_s^{t_n,x}\Delta \tilde{W}_{t_{n+1}}]ds - \Delta t_n \mathbb{E}_{t_n}^{X^n}[f_{t_{n+1}}^{t_n,X^n}\Delta \tilde{W}_{t_{n+1}}] \right|^2 \leq C\left(1 + \mathbb{E}_{t_n}^{X^n}[|X^n|^8]\right)(\Delta t_n)^6, \\
|R_{z2}^n| &= \left| \frac{1}{2}\Delta t_n Z_{t_n}^{t_n,X^n} - \mathbb{E}_{t_n}^{X^n}\left[\left(\int_{t_n}^{t_{n+1}} Z_s^{t_n,x} dW_s\right)\Delta \tilde{W}_{t_{n+1}}\right] \right|^2 \leq C\left(1 + \mathbb{E}_{t_n}^{X^n}\left[|X^n|^8\right]\right)(\Delta t_n)^6, \\
|R_z^n|^2 &\leq C\left(1 + \mathbb{E}_{t_n}^{X^n}\left[|X^n|^8\right]\right)(\Delta t)^6 ,
\end{aligned} \tag{3.22}$$

where C is a positive constant depending on T, K and the upper bounds of the derivatives of b, σ, f and φ.

The proofs of Lemmas 3.2 and 3.3 are quite similar to those for certain lemmas in Refs. [18], so omitted here.

Let us now proceed to deduce the error estimate for Scheme 1. We state the error estimate result in the following theorem.

Theorem 3.2. *Under Assumption* 3.2 *and the conditions of Lemma* 3.2*, for sufficiently small time step* Δt*, we have the estimate*

$$
\begin{aligned}
&\mathbb{E}\left[|e_y^n|^2\right]+\Delta t\sum_{i=n}^{N-1}(1+C\Delta t)^{i-n}\mathbb{E}\left[|e_z^i|^2\right]\\
\leq &C_1\left(\mathbb{E}[|e_y^N|^2]+\Delta t\mathbb{E}[|e_z^N|^2]\right)+C_2\left(\Delta t^{2\beta}+\Delta t^{2\gamma}+\Delta t^4\right)
\end{aligned}
$$

for $0\leq n\leq N-1$*, where* C *is a positive constant depending on* c_0 *and* L*,* C_1 *is a positive constant depending on* c_0*,* T *and* L*,* C_2 *is also a positive constant depending on* c_0*,* T*,* L*,* K*, the initial value of* X_t *in* (1.1)*, and the upper bounds of the derivatives of* b*,* σ*,* f *and* φ*.*

Proof. From the definitions of R_{y1}^i, R_{y2}^i, R_{y3}^i, R_{z1}^i and R_{z2}^i in Theorem 3.1, if Assumption 3.2 holds we have the following estimates under the conditions of Lemma 3.2:

$$
\begin{aligned}
&\mathbb{E}\left[|X^i|^2\right]\leq C\left(1+\mathbb{E}[|X_0|^2]\right),\\
&\mathbb{E}\left[|R_{y_1}^i|^2\right]\leq C\left(1+\mathbb{E}[|X^i|^{4r_1}]\right)(\Delta t)^{2\beta+2}\leq C\left(1+\mathbb{E}[|X_0|^{4r_1}]\right)(\Delta t)^{2\beta+2},\\
&\mathbb{E}\left[|R_{y_2}^i|^2\right]\leq C\left(1+\mathbb{E}[|X^i|^{4r_2}]\right)(\Delta t)^{2\beta+2}\leq C\left(1+\mathbb{E}[|X_0|^{4r_2}]\right)(\Delta t)^{2\beta+2},\\
&\mathbb{E}\left[|R_{y_3}^i|^2\right]\leq C\left(1+\mathbb{E}[|X^i|^{4r_3}]\right)(\Delta t)^{2\beta+2}\leq C\left(1+\mathbb{E}[|X_0|^{4r_3}]\right)(\Delta t)^{2\beta+2},\\
&\mathbb{E}\left[|R_{z_1}^i|^2\right]\leq C\left(1+\mathbb{E}[|X^i|^{4r_4}\right)(\Delta t)^{2\gamma+2}\leq C\left(1+\mathbb{E}[|X_0|^{4r_4}\right)(\Delta t)^{2\gamma+2},\\
&\mathbb{E}\left[|R_{z_2}^i|^2\right]\leq C\left(1+\mathbb{E}[|X^i|^{4r_5}]\right)(\Delta t)^{2\gamma+2}\leq C\left(1+\mathbb{E}[|X_0|^{4r_5}]\right)(\Delta t)^{2\gamma+2}
\end{aligned}\tag{3.23}
$$

for $i=0,1,\cdots,N-1$, where $r_i(i=1,\cdots,5)$ are positive numbers independent of the time partition. From Lemma 3.2, for $0\leq i\leq N-1$ we have

$$
\mathbb{E}\left[|R_y^i|^2\right]\leq C\left(1+\mathbb{E}[|X_0|^8]\right)(\Delta t)^6,\qquad \mathbb{E}\left[|R_z^i|^2\right]\leq C\left(1+\mathbb{E}[|X_0|^8]\right)(\Delta t)^6.\tag{3.24}
$$

From (3.23) and (3.24),

$$
\begin{aligned}
&\sum_{i=n}^{N-1}(1+C\Delta t)^{i-n}\frac{C\mathbb{E}\left[|R_{y_1}^i|^2+(\Delta t)^2(|R_{y_2}^i|^2+|R_{y_3}^i|^2)+|R_y^i|^2\right]}{\Delta t}\\
\leq &C\left(1+\sum_{i=1}^{3}\mathbb{E}[|X_0|^{4r_i}]+\mathbb{E}[|X_0|^8]\right)\left((\Delta t)^{2\beta}+(\Delta t)^4\right)
\end{aligned}\tag{3.25}
$$

and

$$
\begin{aligned}
&\sum_{i=n}^{N-1}(1+C\Delta t)^{i-n}\Delta t\mathbb{E}\left[\left(\frac{2}{\Delta t_n}\right)^2|R_{z_1}^i|^2+|R_{z_2}^i|^2+\left(\frac{2}{\Delta t_n}\right)^2|R_z^i|^2\right]\\
\leq &C\left(1+\mathbb{E}[|X_0|^{4r_4}]+\mathbb{E}[|X_0|^{4r_5}]+\mathbb{E}[|X_0|^8]\right)\left((\Delta t)^{2\gamma}+(\Delta t)^4\right).
\end{aligned}\tag{3.26}
$$

From Theorem 3.1, and the estimates (3.25) and (3.26), we complete the proof. □

Remark 3.3. Under Theorem 3.2, the accuracy of Scheme 1 depends on the accuracy of the numerical methods used to solve the forward SDE. The explicit scheme has first-order accuracy in solving for Y_t and Z_t when either the Euler scheme or the Milstein scheme is used, but it is second-order accuracy when weak order-2.0 Itô-Taylor schemes are used.

4. Numerical Experiments

Some numerical calculations have been carried out to illustrate the high accuracy of our explicit scheme. To confirm the conclusion of Theorem 3.2, we used the Euler scheme, the Milstein scheme and a weak order-2.0 Itô-Taylor (Weak 2.0 for short) scheme to solve the forward SDE. The Weak 2.0 scheme chosen to solve the forward SDE was [6]

$$\begin{aligned} X^{n+1} =& X^n + b^n\Delta t_n + \sigma^n \Delta W_{n,t_{n+1}} + \frac{1}{2}\sigma^n\sigma_x^n\left(\Delta W_{n,t_{n+1}}^2 - \Delta t_n\right) \\ &+ \frac{1}{2}\left[\sigma_t^n + \sigma^n b_x^n + b^n\sigma_x^n + \frac{1}{2}(\sigma^n)^2\sigma_{xx}^n\right]\Delta t_n \Delta W_{n,t_{n+1}} \\ &+ \frac{1}{2}\left[b_t^n + b^n b_x^n + (\sigma^n)^2 b_{xx}^n\right](\Delta t_n)^2 . \end{aligned} \tag{4.1}$$

All three numerical schemes satisfy Assumption 3.2 — cf. [18]. For the Euler and Milstein schemes, the inequalities (3.17)-(3.19) hold for $\beta = \gamma = 1.0$, while for the weak order-2.0 scheme they hold for $\beta = \gamma = 2.0$.

In our numerical experiments, we set $X_0 = 0$, the initial time $t_0 = 0$ and the terminal time $T = 1.0$. The degree of the Lagrangian interpolation polynomials was $k = 5$. In the following tables, $|Y_0 - Y^0|$ and $|Z_0 - Z^0|$ denote the absolute errors between the exact and numerical solutions for Y_t and Z_t at (t_0, X_0), respectively (CR stand for convergence rate).

Example 4.1. In this example, we test Scheme 1 for the following decoupled FBSDE with a linear driver function f with respect to Y_t and Z_t:

$$\begin{cases} X_t =X_0 + \displaystyle\int_0^t \sin(s+X_s)\mathrm{d}s + \int_0^t \cos(s+X_s)\mathrm{d}W_s, \\ Y_t = \sin(T+X_T)\cos(t+X_T) + \displaystyle\int_t^T \Big[2Y_s\cos^2(s+X_s) \\ \qquad + \Big(\sin^2(s+X_s) - \cos^2(s+X_s)\Big)\Big(1+\sin(s+X_s)+\cos^2(s+X_s)\Big) \\ \qquad + Z_s\cos(s+X_s)\Big]\mathrm{d}s - \displaystyle\int_t^T Z_s\mathrm{d}W_s . \end{cases} \tag{4.2}$$

The exact solution of (4.2) is

$$\begin{cases} Y_t = \sin(t+X_t)\cos(t+X_t), \\ Z_t = \cos(t+X_t)\Big(\cos^2(t+X_t) - \sin^2(t+X_t)\Big) . \end{cases} \tag{4.3}$$

The errors and convergence rates for different time partitions and different SDE schemes (the Euler, the Milstein, and the Weak 2.0 schemes) are listed in Table 1.

Table 1: Errors and convergence rates for Example 4.1.

SDE Scheme	Euler		Milstein		Weak 2.0	
N	$\lvert Y_0-Y^0\rvert$	$\lvert Z_0-Z^0\rvert$	$\lvert Y_0-Y^0\rvert$	$\lvert Z_0-Z^0\rvert$	$\lvert Y_0-Y^0\rvert$	$\lvert Z_0-Z^0\rvert$
16	8.689E-02	1.877E-02	6.485E-02	1.403E-02	4.690E-03	1.550E-03
32	4.528E-02	1.201E-02	3.296E-02	9.593E-03	1.226E-03	3.632E-04
64	2.308E-02	6.653E-03	1.661E-02	5.466E-03	3.140E-04	7.851E-05
128	1.165E-02	3.491E-03	8.337E-03	2.903E-03	7.947E-05	1.924E-05
256	5.850E-03	1.786E-03	4.176E-03	1.493E-03	1.998E-05	4.429E-06
CR	0.974	0.857	0.990	0.819	1.970	2.114

Table 2: Errors and convergence rates for Example 4.2.

SDE Scheme	Euler		Milstein		Weak 2.0	
N	$\lvert Y_0-Y^0\rvert$	$\lvert Z_0-Z^0\rvert$	$\lvert Y_0-Y^0\rvert$	$\lvert Z_0-Z^0\rvert$	$\lvert Y_0-Y^0\rvert$	$\lvert Z_0-Z^0\rvert$
16	1.413E-03	1.695E-03	1.276E-03	1.662E-03	7.467E-05	1.291E-04
32	7.158E-04	8.023E-04	6.465E-04	7.859E-04	1.860E-05	3.440E-05
64	3.600E-04	3.893E-04	3.252E-04	3.811E-04	4.713E-06	8.895E-06
128	1.806E-04	1.916E-04	1.632E-04	1.875E-04	1.158E-06	2.240E-06
256	9.045E-05	9.502E-05	8.171E-05	9.295E-05	2.812E-07	5.578E-07
CR	0.992	1.038	0.992	1.039	2.011	1.965

Example 4.2. This is an FBSDE example with a nonlinear driver function:

$$\begin{cases} X_t = X_0 + \displaystyle\int_0^t \frac{1}{1+2\exp(s+X_s)}\mathrm{d}s + \int_0^t \frac{\exp(s+X_s)}{1+\exp(s+X_s)}\mathrm{d}W_s\,, \\ Y_t = \dfrac{\exp(t+X_T)}{1+\exp(t+X_T)} + \displaystyle\int_t^T \Big(-\frac{2Y_s}{1+2\exp(s+X_s)} \\ \qquad -\dfrac{1}{2}\Big(\dfrac{Y_s Z_s}{1+\exp(s+X_s)} - Y_s^2 Z_s\Big)\Big)\mathrm{d}s - \displaystyle\int_t^T Z_s \mathrm{d}W_s\,. \end{cases} \tag{4.4}$$

The analytic solution of (4.4) is

$$\begin{cases} Y_t = \dfrac{\exp(t+X_t)}{1+\exp(t+X_t)}, \\ Z_t = \dfrac{(\exp(t+X_t))^2}{(1+\exp(t+X_t))^3}\,. \end{cases} \tag{4.5}$$

The errors and convergence rates of our experiments are listed in Table 2.

As shown in Tables 1 and 2, for the weak order-1.0 schemes such as the Euler scheme or the Milstein scheme the convergence rate of our explicit scheme is only 1 — but when we used the weak order-2.0 scheme, the convergence rate was 2. This outcome is consistent with Theorem 3.2.

5. Conclusions

An explicit numerical scheme for solving decoupled forward-backward stochastic differential equations is proposed in this article. Errors for the scheme have been analysed, and a general error estimate was obtained for decoupled forward-backward stochastic differential equations that also guarantees the scheme is stable. Under some regularity conditions on the coefficients of the decoupled forward-backward stochastic differential equations, the error estimate implies that the accuracy of the explicit scheme depends on the numerical method for solving forward SDE. In particular, the proposed method is generally order-1 accurate for solving Y_t and Z_t when the Euler or Milstein scheme is used, and order-2 accurate when weak order-2.0 Itô-Taylor type schemes are used.

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