East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.100115.170615a

Vol. **5**, No. 4, pp. 312-326
November 2015

# Backward Error Analysis for Eigenproblems Involving Conjugate Symplectic Matrices

Wei-wei Xu[1,*], Wen Li[2] and Xiao-qing Jin[3]

[1] *School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, P.R. China.*
[2] *School of Mathematical Sciences, South China Normal University, Guangzhou, 510631, P.R. China.*
[3] *Department of Mathematics, University of Macau, Macau.*

*Received 10 January 2015; Accepted (in revised version) 17 June 2015*

**Abstract.** Conjugate symplectic eigenvalue problems arise in solving discrete linear-quadratic optimal control problems and discrete algebraic Riccati equations. In this article, backward errors of approximate pairs of conjugate symplectic matrices are obtained from their properties. Several numerical examples are given to illustrate the results.

**AMS subject classifications**: 65F10, 65F15, 65H10, 65L15.

**Key words**: Backward error, approximate eigenpairs, conjugate symplectic matrix.

## 1. Introduction

An important conjugate symplectic eigenvalue problem that arises in the solution of discrete linear-quadratic optimal control problems and discrete algebraic Riccati equations is formulated as follows [18–21]. Let $\mathbb{C}^{m\times n}$ be the set of $m\times n$ complex matrices. A matrix $A\in\mathbb{C}^{m\times m}$ is said to be conjugate symplectic if $m=2n$ and $A^*JA=J$, where

$$J=\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Numerical algorithms for this eigenproblem have been proposed by many authors [2]. To judge the stability of the computed solution, one may introduce the backward error of an approximate eigenpair $(\mathbf{x},\lambda)$ of the matrix $A$, as a measure of the smallest perturbation $E$ such that

$$(A+E)\mathbf{x}=\lambda\mathbf{x}\,,$$

and by comparing the backward error with the size of any uncertainties in the data matrix $A$ determine whether the pair $(\mathbf{x},\lambda)$ solves the perturbed problem. Let us denote the trace, the

*Corresponding author. *Email address:* `wwx19840904@sina.com` (W. Xu)

 

conjugate, the transpose, the conjugate transformation and the Moore-Penrose generalised inverse of the matrix $A$ by $tr(A)$, $\bar{A}$, $A^T$, $A^*$, and $A^\dagger$, respectively. The identity matrix of order $m$ is denoted by $I_m$, and the Frobenius norm of a matrix is defined by

$$\|A\|_F = \sqrt{tr(A^*A)}\,.$$

We denote by $P_A$ and $P_A^\perp$ the orthogonal projection onto the range $\mathscr{R}(A)$ of the matrix $A$ and the projection complementary to $P_A$, respectively.

The normwise backward error of an eigenpair $(\mathbf{x}, \lambda)$ may be defined as

$$\eta(\mathbf{x}, \lambda) = \min_E \left\{ \alpha^{-1}\|E\|_F : (A+E)\mathbf{x} = \lambda \mathbf{x} \right\}, \tag{1.1}$$

where $\alpha$ is a given positive parameter that allows us some freedom in how the perturbations are measured. Backward errors of approximate solutions are very important, because they can be used to assess the stability and quality of numerical algorithms. Let us denote $X_k = (\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k)$, $\Lambda_k = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_k)$, and the set of approximate eigenpairs by $\{(\mathbf{x}_j, \lambda_j),\ j = 1, \cdots, k\}$. The following definition of the backward error [1]

$$\eta(X_k, \Lambda_k) = \min_E \left\{ \alpha^{-1}\|E\|_F : (A+E)X_k = X_k\Lambda_k \right\}, \tag{1.2}$$

is a natural generalisation of (1.1).

It is notable that this definition is only suitable for unstructured matrix eigenpairs. Since many applications involve structured eigenproblem computation, a definition of the corresponding structured backward error is needed. Thus if $\mathscr{K}$ is a set of some classes of structured matrices, the structured backward error of approximate eigenpairs for the structured matrix class $\mathscr{K}$ may be defined by

$$\eta_{\mathscr{K}}(X_k, \Lambda_k) = \min_E \left\{ \alpha^{-1}\|E\|_F : (A+E)X_k = X_k\Lambda_k,\ A,\ A+E \in \mathscr{K} \right\}. \tag{1.3}$$

From Eqs. (1.2) and (1.3) it is easy to see that

$$\eta(X_k, \Lambda_k) \le \eta_{\mathscr{K}}(X_k, \Lambda_k).$$

Explicit expressions for structured backward error $\eta_{\mathscr{K}}(X_k, \Lambda_k)$ have been given for some particular classes $\mathscr{K}$ — viz.

- the set of unitary matrices [3];
- the set of Hermitian unitary matrices [1];
- the set of symplectic unitary matrices [1];
- the set of symmetric orthogonal matrices [1]; and
- the set of symmetric centrosymmetric and symmetric skew-centrosymmetric matrices, respectively [22].