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A Posteriori Error Estimates of a Weakly Over-Penalized Symmetric Interior Penalty Method for Elliptic Eigenvalue Problems

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Abstract. A weakly over-penalized symmetric interior penalty method is applied to solve elliptic eigenvalue problems. We derive *a posteriori* error estimator of residual type, which proves to be both reliable and efficient in the energy norm. Some numerical tests are provided to confirm our theoretical analysis.

AMS subject classifications: 65N15, 65N25, 65N30.

Key words: Interior penalty method, weakly over-penalization, elliptic eigenvalue problems, *a posteriori* error estimate.

1. Introduction

Adaptive finite element methods for solving partial differential equations based on *a posteriori* error estimation are well developed. *A posteriori* error estimates for eigenvalue problems made impressive progress in recent years — e.g. for conforming finite element methods [16,20,22,23,25,26,30], nonconforming finite element methods [17,31,39], and for mixed finite element methods [19,28]. However, in contrast to the Galerkin methods, *a posteriori* analysis of the discontinuous Galerkin (DG) method for eigenvalue problems is still very rare. In recent years, DG methods have received much attention due to their suitability for *hp*-adaptive techniques and their flexibility in handling highly nonuniform and unstructured meshes. In addition, the DG scheme can easily handle inhomogeneous boundary conditions and curved boundaries. *A posteriori* error bounds for DG methods for solving source problems have been extensively studied in the literature — cf. [1,5,21,27,29,32–34,38] and references therein.

The weakly over-penalized symmetric interior penalty (WOPSIP) DG method was initially proposed by Brenner *et al.* [11] to solve second order elliptic equations. An *a priori*

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error estimate has been provided for application to second order elliptic equations [11], and subsequently a residual-based *a posteriori* error estimator was derived [8]. From Ref. [11] we know that the WOPSIP method has many advantages compared with other well-known DG methods [2], including less computational complexity and its ease of implementation. Moreover, the WOPSIP method has high intrinsic parallelism [9]. For these reasons, the WOPSIP methods have been further developed to solve non-self-adjoint and indefinite problems [42], biharmonic problems [10], Stokes equations [3], Reissner-Mindlin plate equations [6] and variational inequalities [41]. However, the first detailed *a posteriori* analysis of DG methods for eigenvalue problems is quite recent [24], and here we derive a residual-based *a posteriori* estimator for a second order elliptic eigenvalue problem that is both reliable and efficient. In Section 2, we first give some notation and then introduce the WOPSIP method for the model problem. In Section 3, we present our residual-based *a posteriori* error estimator, and demonstrate its reliability and efficiency. Some numerical tests supporting our theoretical analysis are provided in Section 4, and our final remarks appear in Section 5.

2. Problem Set-up and the WOPSIP Method

We consider the following elliptic eigenvalue problem:

$$\begin{aligned} -\Delta u &= \lambda u \quad \text{in } \Omega\,, \\ u &= 0 \quad \text{on } \Gamma\,, \end{aligned} \tag{2.1}$$

where Δ denotes the Laplacian operator and $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain with boundary $\Gamma = \partial\Omega$. Let us now first introduce some notation. For a bounded domain $\mathscr{D}$ in $\mathbb{R}^2$, we denote by $H^s(\mathscr{D})$ the standard Sobolev space of functions with regularity exponent $s \geq 0$, associated with norm $\|\cdot\|_{s,\mathscr{D}}$ and seminorm $|\cdot|_{s,\mathscr{D}}$. When $s = 0$, $H^0(\mathscr{D})$ can be written $L^2(\mathscr{D})$, and we denote the inner product in $L^2(\mathscr{D})$ by $(\cdot,\cdot)_{\mathscr{D}}$. When $\mathscr{D} = \Omega$, the domain subscript is dropped, and we set $V = H_0^1(\Omega) := \{v \in H^1(\Omega) : v|_\Gamma = 0\}$. After normalisation for u, the weak formulation of the eigenvalue problem (2.1) reads:

Find $(\lambda, u) \in \mathbb{R} \times V$ such that $\|u\|_0 = 1$ and

$$a(u,v) = \lambda(u,v) \quad \forall v \in V\,, \tag{2.2}$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$$

and ∇ denotes the gradient operator.

Let $\mathscr{T}_h$ be a shape-regular decomposition of Ω into triangles $\{T\}$. We set $h_T = \operatorname{diam}(T)$ and $h = \max_{T\in\mathscr{T}_h} h_T$, and denote by $\mathscr{E}_h^I$ the set of interior edges of the elements in $\mathscr{T}_h$. The subset of edges on $\partial\Omega$ is denoted by $\mathscr{E}_h^\partial$, so the set of all edges $\mathscr{E}_h = \mathscr{E}_h^I \cup \mathscr{E}_h^\partial$ and h_e is the length of the edge $e \in \mathscr{E}_h$. For a subset $\mathscr{D}$ of $\mathbb{R}^2$, we denote by $P_k(\mathscr{D})$ the space of polynomials of degree less than or equal to k on $\mathscr{D}$. Furthermore, we associate a fixed unit