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A Fully Discrete Spectral Method for Fisher's Equation on the Whole Line

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Abstract. A generalised Hermite spectral method for Fisher's equation in genetics with different asymptotic solution behaviour at infinities is proposed, involving a fully discrete scheme using a second order finite difference approximation in the time. The convergence and stability of the scheme are analysed, and some numerical results demonstrate its efficiency and substantiate our theoretical analysis.

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Key words: Fisher's equation, spectral method, generalised Hermite function, problem on the whole line.

1. Introduction

In recent years, the numerical solution of problems defined on unbounded domains has received increasing attention. Hermite orthogonal approximation and Hermite-Gauss interpolation are appropriate to solve such problems defined on the whole line [1, 2, 4, 8–10, 12–14, 18–23, 26, 28], and many articles discuss solutions that decay at algebraic rates as $|x| \to \infty$ [1, 4, 13, 29]. There are also spectral methods for solutions that decay at a certain algebraic rate as $x \to +\infty$ but at another algebraic rate as $x \to -\infty$ [14, 15, 27]. In this article, and here we investigate a spectral method for the nonlinear partial differential equation in genetics known as Fisher's equation with different asymptotic solution behaviour at infinities.

If $U(x',t')$ denotes population density as a function of length x' and time t', Fisher's equation describing the distribution of an advantageous gene as it diffuses through a given population is

$$\partial_{t'}U(x',t') - D\partial_{x'}^2 U(x',t') = rU(x',t')\left(1 - \frac{U(x',t')}{K}\right), \quad x' \in R, \ 0 < t' \le T, \quad (1.1)$$

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where $r > 0$ is the growth rate, $K > 0$ the carrying capacity, $D > 0$ the constant diffusion coefficient and the right-hand side $f(U) = rU(1-U/K)$ is the assumed local source term given by the Logistics growth law [5,6] . Under the variable transformations [6]

$$t = \frac{t'}{r^{-1}}\,, \qquad x = \frac{x'}{\sqrt{D/r}}\,, \qquad u = \frac{U}{K}\,,$$

Fisher's equation (1.1) may be rewritten as

$$\partial_t u(x,t) - \partial_x^2 u(x,t) = u(x,t) - u^2(x,t)\,. \tag{1.2}$$

Some numerical algorithms were proposed for Fisher's equation (1.2) with specific initial and boundary conditions defined on bounded intervals or on the whole line — cf. Refs. [3,16,17,24] and references therein. When considering a spectral method for Fisher's equation (1.2) with different asymptotic solution behaviour at infinities, we could proceed in one of two ways. One way is to use an orthogonal approximation where the weight function behaves differently at the infinities — cf. Refs. [14] and [15] for example, where the authors considered Eq. (1.2) when the local source term $f(U) = aU(1-U)$ involves the constant $a > 0$ that measures the intensity of insects, and implemented numerical analysis to solve Eq. (1.2) in weighted Sobolev spaces. However, the weight function destroys the symmetry and positive definiteness of the bilinear operators. The second way is to reformulate the original problem to a related homogeneous boundary value problem at infinities or the domain boundaries and then solve that — e.g. a Dirichlet boundary value problem [25] or a Neumann boundary value problem [26] in two-dimensions. Further, it is then appropriate to consider generalised Hermite function approximations with a scaling factor σ and weight function $\chi(x) \equiv 1$, as in the recent numerical simulation of the generalised Ginzburg-Landau equation [29].

Here we obtain an auxiliary function that simulates the different asymptotic solution behaviour of Eq. (1.2), and then we reformulate the underlying problem as an homogeneous boundary value problem with a solution that decays very fast as $x \to \infty$. Following Ref. [29], we adopt generalised Hermite function as the basis in our theoretical analysis and computation, which has several advantages:

1. the auxiliary function $W_B(x) = 1 - \frac{e^x}{1+e^x}$ unifies the different asymptotic behaviour $u(x,t) \to 0,\ x \to +\infty$ and $u(x,t) \to 1,\ x \to -\infty$, which facilitates the theoretical analysis and numerical implementation;
2. the scaled generalised Hermite functions simplify the analysis and produce more precise error estimates; and
3. the orthogonality of the scaled generalised Hermite functions leads to a sparse system in the unknown coefficients of the expansion for $U(x,t)$ (with a diagonal or pentadiagonal coefficient matrix) and the numerical solution has spectral accuracy.

In Section 2, we recall some basic approximation results and inverse inequalities for the generalised Hermite spectral approximation, which play important roles in the numerical analysis of spectral methods for Fisher's equation on the whole line. In Section 3, we