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The Maximal Solution and Comparison Theorems for the Periodic Discrete-Time Riccati Equation

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Abstract. Properties and comparison theorems for the maximal solution of the periodic discrete-time Riccati equation are supplemented by an extension of some earlier results and analysis, for the discrete-time Riccati equation to the periodic case.

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Key words: The maximal solution, pd-stabilisable, p-observable, comparison theorems.

1. Introduction

The following periodic discrete-time Riccati equation (PDRE) is considered:

$$X_{j-1} = A_j^* X_j A_j - (B_j^* X_j A_j + C_j)^* (R_j + B_j^* X_j B_j)^{-1} (B_j^* X_j A_j + C_j) + Q_j, \quad j = 1, \cdots, p, \tag{1.1}$$

where $X_0 = X_p$ and the superscript $*$ denotes the matrix conjugate transpose, and the matrices $A_j \in \mathbb{C}^{n_{j+1}\times n_j}$, $B_j \in \mathbb{C}^{n_{j+1}\times m_j}$, $C_j \in \mathbb{C}^{m_j\times n_j}$, $R_j = R_j^* \in \mathbb{C}^{m_j\times m_j}$, $Q_j = Q_j^* \in \mathbb{C}^{n_j\times n_j}$ have period $p \geq 1$ — i.e. $A_{j+p} = A_j$, $B_{j+p} = B_j$, $C_{j+p} = C_j$, $R_{j+p} = R_j$, $Q_{j+p} = Q_j$. For Hermitian matrices M and N, let $M \geq N$ (respectively, $M > N$) denote that the matrix difference $M - N$ is Hermitian positive semidefinite (respectively, Hermitian positive definite), $Q^{1/2}$ denote the Hermitian positive semidefinite square root of a Hermitian positive semidefinite matrix Q, and throughout assume that $R_j > 0$ for $j = 1, \cdots, p$.

As is well known, the PDRE (1.1) arises when solving the periodic linear-quadratic optimal control problem [4] for the linear discrete-time periodic system

$$\begin{aligned} x_{j+1} &= A_j x_j + B_j u_j, \\ y_j &= S_j x_j, \end{aligned} \tag{1.2}$$

by minimising the quadratic cost functional

$$J = \frac{1}{2}\sum_{j=0}^{\infty} [x_j^*, u_j^*] \begin{bmatrix} Q_j & C_j^* \\ C_j & R_j \end{bmatrix} \begin{bmatrix} x_j \\ u_j \end{bmatrix}, \tag{1.3}$$

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where $Q_j = S_j^* S_j$, and it is usually assumed that

$$\begin{bmatrix} Q_j & C_j^* \\ C_j & R_j \end{bmatrix} \geq 0 \, .$$

The periodic optimal feedback control u_j is given by

$$u_j^{\#} = -(R_j + B_j^* X_j B_j)^{-1}(B_j^* X_j A_j + C_j)x_j \, , \quad j = 1, \cdots, p \, ,$$

where $\{X_j\}_{j=1}^p$ is the Hermitian positive semidefinite solution to the PDRE (1.1). The state transition matrix of the periodic system (1.2) is defined as the $n_j \times n_i$ matrix $\Phi_A(j,i) = A_{j-1}A_{j-2}\cdots A_i$, where $\Phi_A(i,i) = I_{n_i}$ — i.e. the identity matrix of order n_i. The state transition matrix over one whole period $\Phi_A(j+p,j) = A_{j-1}\cdots A_1 A_p \cdots A_j \in \mathbb{C}^{n_j \times n_j}$ is called the monodromy matrix of the system (1.2) at time j, and its eigenvalues are called the characteristic multipliers at time j. From Theorem 1.3.20 in Ref. [8], matrices $\Phi_A(j+p,j)$ have the same nonzero characteristic multipliers for all j. The system (1.2) or $\Phi_A(1+p,1)$ is called asymptotically stable if and only if its characteristic multipliers belong to the open unit disc.

An Hermitian periodic solution $\{X_j^+\}_{j=1}^p$ of the PDRE (1.1) is said to be maximal if $X_j^+ \geq X_j, j = 1, \cdots, p$ for any Hermitian periodic solution $\{X_j\}_{j=1}^p$ of (1.1). An Hermitian periodic solution $\{X_j\}_{j=1}^p$ of the PDRE (1.1) is called stabilising (respectively, strong) if all eigenvalues of $\Phi_{\widehat{A}}(1+p,1)$ are in the open (respectively, closed) unit disc, where $\widehat{A}_j = A_j - B_j(R_j + B_j^* X_j B_j)^{-1}(B_j^* X_j A_j + C_j)$.

There are many articles in the literature on the perturbation theory and numerical methods of the PDRE (1.1) — e.g. see Refs. [10, 12, 14, 18, 21]. The study of periodic Riccati difference equations can be traced back to Ref. [16], which provided existence and uniqueness conditions of periodic stabilising and strong solutions. Bittanti *et al.* [3] and Souza [19,20] provided several theorems on the existence, uniqueness and stability properties of Hermitian periodic positive semidefinite solutions of the PDRE. The maximal solution and comparison theorems for algebraic Riccati equations have also been considered — e.g. see Refs. [9,17,22]). To date there seems to have been no systematic study on the maximal periodic solution and comparison theorems for the PDRE (1.1). Attention is focused on two problems here — viz. the existence and properties of the maximal periodic solution of the PRDE (1.1) and comparison theorems between two different PDRE, leading to extensions of corresponding results in Refs. [9,17,22]. Section 2 is devoted to basic concepts for periodic control systems and eigenproblems of periodic matrix pairs together with other results to be used in Section 3, where the existence of the maximal periodic solution of the PDRE (1.1) and its properties are discussed. Comparison theorems between two different PDRE are presented in Section 4, and conclusions are in Section 5.

2. Preliminaries

Let $A \in \mathbb{C}^{n\times n}, B \in \mathbb{C}^{n\times m}$ and $C \in \mathbb{C}^{m\times n}$. The matrix pair (C,A) is said to be observable if $Cx = 0$ and $Ax = \lambda x$ for any number λ imply $x = 0$. The matrix pair (A,B) is d-stabilisable