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Numerical Analysis for a Nonlocal Parabolic Problem

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Abstract. This article is devoted to the study of the finite element approximation for a nonlocal nonlinear parabolic problem. Using a linearised Crank-Nicolson Galerkin finite element method for a nonlinear reaction-diffusion equation, we establish the convergence and error bound for the fully discrete scheme. Moreover, important results on exponential decay and vanishing of the solutions in finite time are presented. Finally, some numerical simulations are presented to illustrate our theoretical analysis.

AMS subject classifications: 65N30

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1. Introduction

Partial differential equations often arise in mathematical models describing physical, chemical, biological and ecological systems. Chipot & Lovat [1] studied the following nonlocal problem:

$$\begin{cases} u_t - a(l(u))\Delta u = f(x,t) \ \text{ in } \Omega \times (0,T)\,, \\ u(x,t) = 0 \ \text{ on } \partial\Omega \times (0,T)\,, \\ u(x,0) = u_0(x) \ \text{ in } \Omega\,, \end{cases} \tag{1.1}$$

where the domain Ω is a subset of $\mathbb{R}^d$, $d \geq 1$ with a smooth boundary $\partial\Omega$, T is an arbitrary time, $a(l(u))$ is some function from $\mathbb{R}$ to $(0,\infty)$, and Δ denotes the Laplacian operator. For instance, this problem describes the density u of a population (e.g. bacteria) subject to spreading, where the diffusion coefficient a may depend upon the entire population in the domain rather than the local density. The model is said to be nonlocal if the evolution is guided by the global state, as it is in that case [2]. Further applications of this model are discussed in Refs. [3,4] and references therein.

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Chipot & Lovat [1] proved the existence and uniqueness of the solution to the problem (1.1), and Correa *et al.* [5] extended their analysis on assuming that $a(l(u))$ and $f(x,u)$ are continuous functions. However, there are few studies on the numerical solution of nonlocal problems, and they are restricted to nonlocal reaction terms or nonlocal boundary conditions. Yin & Xu [7] applied the finite volume method to obtain approximate solutions for a nonlocal problem on reactive flows in porous media, and derived the optimal convergence order in the L^2 norm. Sidi Ammi & Torres [8] applied the finite element method to the space variables and the Euler or Crank-Nicolson method to the time, in fully discretising a nonlocal thermistor problem. They also proved optimal rates of convergence in the L^2 norm. Almeida *et al.* [9] presented convergence analysis for a fully discretised approximation to a nonlocal problem involving a parabolic equation with moving boundaries, with the finite element method applied for the space variables and the Crank–Nicolson method for the time.

We consider the following parabolic problem with nonlocal nonlinearity:

$$\begin{cases} u_t - a(l(u))\Delta u + \alpha|u|^{p-2}u = f(u) \quad \text{in } \Omega \times (0,T]\,, \\ u(x,t) = 0 \ \text{ on } \partial\Omega \times (0,T]\,, \\ u(x,0) = u_0(x) \ \text{ in } \Omega\,, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^d$, $d \geq 1$ is again a domain with a smooth boundary $\partial\Omega$, a and f are functions to be defined in the next section, and $l : L^2(\Omega) \to \mathbb{R}$ is a continuous linear form. Simsen & Ferreira [6] have discussed not only the existence and uniqueness of solutions for this problem, but also continuity with respect to initial values, the exponential stability of weak solutions, and important results on the existence of a global attractor. Our goal here for this nonlocal reaction–diffusion problem (1.2) is to check the asymptotic behaviour of the solution for large time t numerically, and prove the convergence of a fully discrete approximation using the Crank-Nicolson-Galerkin finite element method.

In Section 2, we discuss hypotheses on the data and the weak variational formulation of the problem, followed by consideration of the asymptotic behaviour of the solution for large time t. Section 3 is concerned with the space-discrete problem and its convergence, and the convergence of the discrete solution is proven in Section 4. Finally, numerical results are presented in Section 5 to illustrate our theoretical analysis.

2. Preliminaries, Weak Formulation and Asymptotic Solution Behaviour

Throughout this article, $H^k(\Omega)$ denotes the usual Sobolev space of order $k \in \mathbb{N}$ with norm $\|\cdot\|_k$ and $H_0^k(\Omega)$ the closure of $\mathscr{C}_0^\infty(\Omega)$ in $H^k(\Omega)$. The Lebesgue space is denoted by $L^r(\Omega)$, $1 < r \leq \infty$ with norm $\|\cdot\|_{L^r}$ that is simplified to $\|\cdot\|$ in $L^2(\Omega)$, and $\mathscr{D}'(\Omega)$ is the space of distributions on Ω. We also employ the standard notation of Bochner spaces, such as $L^q(0,T,X)$ with norm $\|\cdot\|_{L^q(X)}$ where X is an Hilbert space, and denote the potential energy associated with the problem (1.2) by

$$E(t) = \frac{1}{2}\|u(t)\|^2\,.$$