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High Order Well-Balanced Weighted Compact Nonlinear Schemes for the Gas Dynamic Equations under Gravitational Fields

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Abstract. In this study, we propose a high order well-balanced weighted compact nonlinear (WCN) scheme for the gas dynamic equations under gravitational fields. The proposed scheme is an extension of the high order WCN schemes developed in (S. Zhang, S. Jiang, C.-W Shu, J. Comput. Phys. 227 (2008) 7294-7321). For the purpose of maintaining the exact steady state solution, the well-balanced technique in (Y. Xing, C.-W Shu, J. Sci. Comput. 54 (2013) 645-662) is employed to split the source term into two terms. The proposed scheme can maintain the isothermal equilibrium solution exactly, genuine high order accuracy and resolve small perturbations of the hydrostatic balance state on the coarse meshes. Furthermore, in order to capture the strong discontinuities and large gradients, the fifth-order upwind weighted nonlinear interpolations together with the fourth/sixth order cell-centered compact schemes with local characteristic projections are used to construct different WCN schemes. Several representative one- and two-dimensional examples are simulated to demonstrate the good performance of the proposed schemes.

AMS subject classifications: 35L65, 35L67

Key words: Euler equations, gravitational fields, source term, steady state solution, weighted compact nonlinear Scheme.

1. Introduction

Gas dynamic equations under gravitational fields are important in describing the astrophysical and atmospheric phenomena such as supernova explosions [14], and climate modeling and weather forecasting [4]. Since many astrophysical problems involving the hydrodynamical evolution in a gravitational field, correctly capturing the effect of gravitational force plays an important role in the long time simulations of star and galaxy formation. Failure on such capturing would lead to a solution which either oscillates around the equilibrium, or deviates from the equilibrium after a long time simulation.

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Therefore, a reliable numerical scheme for this problem should maintain the precise balance of pressure and gravitational forces in case of the hydrostatic solution. The well-balanced schemes can preserve exactly these steady state solutions up to machine accuracy. They are also specially designed to ensure accurate simulations and exhibit essential stability properties on relatively coarse meshes. In the last two decades, several well-balanced pioneering works have been designed. For example, LeVeque and Bale [16] extended the quasi-steady wave-propagation methods for the Euler equations under a static gravitational field. In [22, 31], Xu and his collaborators designed the well-balanced gas-kinetic scheme for multidimensional gas dynamic equations under gravitational field. The well-balanced Runge-Kutta discontinuous Galerkin method [18] and nodal discontinuous Galerkin method [6] for the Euler equations under gravitational fields can preserve the isothermal hydrostatic solution up to machine precision. A second-order well-balanced central-upwind scheme for the Euler equations of gas dynamics with gravitation was proposed in [7]. An approximate Riemann solver using the formalism of Harten, Lax and van Leer was developed to derive a well-balanced numerical scheme to obtain the solutions of the Euler equations with a gravitational potential in [11]. High order Weighted Essentially Non-Oscillation (WENO) finite difference well-balanced scheme for the isothermal equilibrium was introduced in [33]. High order well-balanced finite volume WENO schemes preserving not only the isothermal equilibrium but also the polytropic hydrostatic balance state exactly was proposed in [19]. Another popular hyperbolic balance laws with source terms in the literature is the shallow water equations over a non-flat bottom topology which also need the well-balanced methods to maintain a delicate balance of the divergence of the flux and the source terms in the steady state solution, such as [1,2,16,20,27,32,36,37,39] and the references therein.

Based on the idea of WENO scheme, a class of weighted compact nonlinear (WCN) schemes were developed [9, 10] which has better wave resolution and similar ability to capture discontinuities as the classical WENO scheme. Then it has been extended to the ninth order [23, 26, 38] which demonstrates the increasing resolutions with the increasing orders. The WCN schemes also show suitable freestream and vortex preservation properties on a wavy grid [25]. In order to preserve the positivity of both density and pressure in the complex flow problems with severe discontinuities, the robust WCN schemes were proposed in [8, 21, 24] respectively. The corresponding robust WCN scheme was successfully employed to simulate the hydrogen/air detonation [24]. Recently, the high order well-balanced WCN schemes for shallow water equations were designed in [12].

In this paper, a numerical framework of the high order well-balanced WCN schemes is designed for the one- and two-dimensional gas dynamic equations under gravitational fields. It is an extension of the high order WCN scheme [38] and well-balanced WENO scheme [33]. The proposed schemes can be proved that the isothermal equilibrium solution and genuine high order accuracy can be exactly maintained by rigorous theoretical analysis. In the numerical simulations, the first order global Lax-Friedrichs flux splitting is used to introduce correct upwinding and enhance the numerical stability. To reach the high order accuracy, the fifth-order upwind weighted nonlinear interpolations together with the fourth/sixth order cell-centered compact schemes are employed. To further restrain the