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A Convergence Analysis of the MINRES Method for Some Hermitian Indefinite Systems

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Abstract. Some convergence bounds of the **min**imal **res**idual (MINRES) method are studied when the method is applied for solving Hermitian indefinite linear systems. The matrices of these linear systems are supposed to have some properties so that their spectra are all clustered around ± 1. New convergence bounds depending on the spectrum of the coefficient matrix are presented. Some numerical experiments are shown to demonstrate our theoretical results.

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Key words: MINRES, Convergence bound, Hermitian indefinite, Toeplitz system.

1. Introduction

Let $H_n \in \mathbb{C}^{n\times n}$ be a nonsingular Hermitian matrix. We consider solving the linear system

$$H_n \mathbf{x}_n = \mathbf{b}_n \tag{1.1}$$

by using the **min**imal **res**idual (MINRES) method, which has favorable properties (see, e.g., [27]):

(i) minimization of the residual;
(ii) short-term recurrences;
(iii) sharp convergence bounds depending on the spectrum of the matrix.

The MINRES method is a Krylov subspace method developed by Paige and Saunders [26]. Thanks to its favorable properties above, it is popular for solving Hermitian linear systems. A large amount of literature (see, e.g., [1–3, 5–7, 9–18, 23–25, 27–30, 33–36]

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and the references therein) exists on discussing the convergence rate of Krylov subspace methods, including the **c**onjugate **g**radient (CG) method, the **g**eneralized **m**inimal **res**idual (GMRES) method, the MINRES method, and so on.

In this paper, we consider the convergence rate of the MINRES method for solving Hermitian indefinite linear systems. Existing convergence analyses for the GMRES and MINRES methods can be found in [16,23,28,29,34] and the references therein. Different from these analyses, our motivation is to demonstrate the relation between a matrix decomposition and the superlinear convergence rate. We try to expose some convergence results for the MINRES method as those in [18, Subsection 1.3] for the CG method. Similar to the case of the CG method, the convergence bounds of the MINRES progress are largely related to the spectral distribution of the coefficient matrix. By using preconditioners and Proposition 2 in [27], we show that the MINRES method has a superlinear convergence rate when it is applied for solving a large class of Toeplitz systems.

2. Preliminaries

Denote the Euclidean norm of vectors by $\|\cdot\|_2$. Let $\mathbf{r}_n^{(k)}$ be the residual vector after kth iteration of the MINRES method applied for solving the Hermitian linear system $H_n\mathbf{x}_n = \mathbf{b}_n$, i.e., $\mathbf{r}_n^{(k)} = \mathbf{b}_n - H_n\mathbf{x}_n^{(k)}$.

Let $\mathscr{P}_k$ be the set of all real polynomials of degree not exceeding k with constant term 1, and $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ (or $\lambda(H_n)$) be the spectrum of H_n. The following results can be found in [15, Subsection 3.1] and [12, Section 6].

Lemma 2.1 ([15]). *Let $\mathbf{r}_n^{(k)}$ be the residual vector after kth iteration of the MINRES method applied to the Hermitian linear system $H_n\mathbf{x}_n = \mathbf{b}_n$. Then one has*

$$\frac{\|\mathbf{r}_n^{(k)}\|_2}{\|\mathbf{r}_n^{(0)}\|_2} \le \min_{p_k \in \mathscr{P}_k} \max_{1\le j\le n} \left|p_k(\lambda_j)\right|. \tag{2.1}$$

Lemma 2.2 ([15]). *Let b_1, b_2, b_3, and b_4 be positive constants with $b_1 - b_2 = b_4 - b_3$. Then*

$$\min_{p_k \in \mathscr{P}_k} \max_{\lambda \in [-b_1,-b_2]\cup[b_3,b_4]} \left|p_k(\lambda)\right| \le 2\left(\frac{\sqrt{b_1b_4} - \sqrt{b_2b_3}}{\sqrt{b_1b_4} + \sqrt{b_2b_3}}\right)^{[k/2]}, \tag{2.2}$$

where $[x]$ denotes the largest integer less than or equal to a number x.

3. Main Theorems

In this section we consider some convergence bounds of the MINRES method when it is applied for solving the Hermitian indefinite linear system $H_n\mathbf{x}_n = \mathbf{b}_n$.

Theorem 3.1. *If the eigenvalues of $H_n \in \mathbb{C}^{n\times n}$ are ordered such that*

$$\lambda_1 \le \cdots \le \lambda_p \le -b_1 \le \lambda_{p+1} \le \cdots \le \lambda_{n_1-q} \le -b_2 \le \lambda_{n_1-q+1} \le \cdots \le \lambda_{n_1} < 0,$$