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Nodal-Type Newton-Cotes Rules for Fractional Hypersingular Integrals

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Abstract. Nodal-type Newton-Cotes rules for fractional hypersingular integrals based on the piecewise k-th order Newton interpolations are proposed. A general error estimate is first derived on quasi-uniform meshes and then we show that the even-order rules exhibit the superconvergence phenomenon — i.e. if the singular point is far away from the endpoints then the accuracy of the method is one order higher than the general estimate. Numerical experiments confirm the theoretical results.

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Key words: Hypersingular integrals, fractional order, nodal-type Newton-Cotes rules, superconvergence.

1. Introduction

Considering the integral

$$\mathscr{I}u(x) = \int_a^b \frac{u(y)}{|y-x|^{1+2s}}\,dy, \quad s \in [0,1), \quad x \in (a,b), \tag{1.1}$$

we note that it does not exist in usual sense and should be specifically defined. These types of integrals are often referred to as Hadamard finite-part integrals or hypersingular integrals. There are various definitions and we first consider the case where the singular point is located at an interval end — cf. [22]. In this case the integral can be defined as

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$$
\begin{aligned}
&\fint_a^x \frac{u(y)}{(x-y)^{1+2s}}\,dy := \lim_{\epsilon\to 0}\left(\int_a^{x-\epsilon} \frac{u(y)}{(x-y)^{1+2s}}\,dy + r_-(\epsilon)\right),\\
&\fint_x^b \frac{u(y)}{(y-x)^{1+2s}}\,dy := \lim_{\epsilon\to 0}\left(\int_{x+\epsilon}^{b} \frac{u(y)}{(y-x)^{1+2s}}\,dy + r_+(\epsilon)\right),
\end{aligned}
\tag{1.2}
$$

where

$$
r_-(\epsilon) = \begin{cases}
u(x_-)\ln\epsilon, & s=0,\\
u(\_)\dfrac{\epsilon^{-2s}}{-2s}, & s\in(0,1/2),\\
-u(x_-)\epsilon^{-1} - u'(x_-)\ln\epsilon, & s=1/2,\\
u(x_-)\dfrac{\epsilon^{-2s}}{-2s} - u'(x_-)\dfrac{\epsilon^{1-2s}}{1-2s}, & s\in(1/2,1),
\end{cases}
$$

$$
r_+(\epsilon) = \begin{cases}
u(x_+)\ln\epsilon, & s=0,\\
u(x_+)\dfrac{\epsilon^{-2s}}{-2s}, & s\in(0,1/2),\\
-u(x_+)\epsilon^{-1} + u'(x_+)\ln\epsilon, & s=1/2,\\
u(x_+)\dfrac{\epsilon^{-2s}}{-2s} + u'(x_+)\dfrac{\epsilon^{1-2s}}{1-2s}, & s\in(1/2,1),
\end{cases}
$$

and $u(x_-)$ and $u(x_+)$ are, respectively, the left and right limits of u at x. If $x\in(a,b)$, then we define the corresponding integral as

$$
\mathscr{I}u(x) := \lim_{\epsilon\to 0}\left[\left(\int_a^{x-\epsilon} + \int_{x+\epsilon}^{b}\right)\frac{u(y)}{|y-x|^{1+2s}}\,dy + r(\epsilon)\right], \quad x\in(a,b), \tag{1.3}
$$

where

$$
r(\epsilon) = r_-(\epsilon) + r_+(\epsilon).
$$

A function $u(y)$ is said to be Hadamard finite-part integrable with respect to the weight $|y-x|^{-1-2s}$ if the limit in the right-hand side of (1.3) exists. It is worth noting that if $u(y)$ has a strong regularity, then $r(\epsilon)$ can be represented as

$$
r(\epsilon) = u(x)\begin{cases}
2\ln\epsilon, & s=0,\\
-\dfrac{\epsilon^{-2s}}{s}, & s\in(0,1).
\end{cases}
$$

The approximation of hypersingular integrals plays an important role in numerical methods for various integral equations arising in acoustics [27], electromagnetics [20,26], heat conduction [18]. Besides, equations with hypersingular integrals are also used in stress calculation [3,9], fracture mechanics [1,2,4,8] and wave scattering [2,11,12]. A special attention has been paid to quadrature formulas for hypersingular integrals, including Gaussian