East Asian Journal on Applied Mathematics Vol. **8**, No. 4, pp. 715-745
doi: 10.4208/eajam.220418.300618 November 2018

A Conservative Difference Scheme for Space Fractional Klein-Gordon-Schrödinger Equations with a High-Degree Yukawa Interaction

Junjie Wang[1,2], Aiguo Xiao[1,*] and Chenxi Wang[1]

[1]*School of Mathematics and Computational Science & Hunan Key Laboratory for Computation and Simulation in Science and Engineering, Xiangtan University, Hunan 411105, China.*
[2]*School of Mathematics and Statistics, Pu'er University, Yunnan 665000, China.*

Received 22 April 2018; Accepted (in revised version) 30 June 2018.

Abstract. A conservative finite difference scheme for nonlinear space fractional Klein-Gordon-Schrödinger systems with high-degree Yukawa interaction is studied. We show that the arising difference equations are uniquely solvable and approximate solutions converge to the exact solution at the rate $\mathscr{O}(\tau^2+h^2)$. Moreover, we prove that the scheme can be decoupled and preserves the mass and energy conservation laws. Numerous examples confirm theoretical results and demonstrate the efficiency of the scheme. They also show the influence of the fractional order and the high-degree term coefficient on the shape and the propagation velocity of solitary waves.

AMS subject classifications: 65M06, 65M12, 35R11

Key words: Space fractional Klein-Gordon-Schrödinger equation, conservative difference scheme, convergence, quantum subdiffusion, local high oscillation.

1. Introduction

The classical system of coupled Klein-Gordon-Schrödinger (KGS) equations with Yukawa interaction has the form

$$\begin{aligned} &iu_t + \Delta u + gu\phi = 0, \\ &\phi_{tt} - \Delta\phi + \mu^2\phi - g|u|^2 = 0, \end{aligned} \tag{1.1}$$

where u and ϕ are, respectively, complex scalar nucleon and real scalar meson fields [11]. The real constant μ is the meson mass and g a coupling constant. The unique solvability of the system (1.1) has been studied in [15, 36], exact solitary wave solutions in [51, 52], stability of stationary states in [26], and attractors of KGS systems in [3, 12]. Moreover,

*Corresponding author. *Email address:* `xag@xtu.edu.cn` (A. Xiao)

in order to solve KGS systems with low-degree Yukawa interaction, various approximation approaches, including spectral [1, 6, 53], symplectic [21, 45], multisymplectic [16], meshless [5], conservative finite differences [29,42,55] and conservative orthogonal spline collocation [50] methods have been used.

On the other hand, fractional calculus is becoming an important tool in applications — cf. Refs. [4, 7, 8, 18, 22, 24, 25, 32, 35, 54]. In particular, space fractional Schrödinger equations describe various physical phenomena [13, 30, 39, 41]. They are solved by conservative difference methods [33, 43, 44, 47–49], mass-conservative Fourier spectral methods [9], fourth-order methods [19, 56], a collocation method [2], a conservative finite element method [23], and HSS-like iteration method [31, 32]. The stability and convergence of these methods have been also investigated.

The space fractional KGS systems play an important role in fractional quantum field theory [34, 40] but the study of such models is mainly restricted to qualitative analysis such as local or global well-posedness [17] and there are only a few works, where numerical approaches are discussed — cf. [46].

Let $(-\Delta)^{\alpha/2}$ refer to the fractional Laplacian operator

$$-(-\Delta)^{\alpha/2}u(x,t) := -\mathscr{F}^{-1}(|\xi|^{\alpha}\mathscr{F}(u(\xi,t))),$$

where $\mathscr{F}$ is the Fourier transform. It can be represented [30] as the Riesz fractional derivative — i.e.

$$-(-\Delta)^{\alpha/2}u(x,t) := \frac{\partial^{\alpha}}{\partial|x|^{\alpha}}u(x,t) = -\frac{1}{2\cos(\alpha\pi/2)}\left[{}_{-\infty}D_x^{\alpha}u(x,t) + {}_xD_{+\infty}^{\alpha}u(x,t)\right],$$

where ${}_{-\infty}D_x^{\alpha}u(x,t)$, ${}_xD_{+\infty}^{\alpha}u(x,t)$ are, respectively, the left and right Riemann-Liouville fractional derivatives of order α — i.e.

$$
\begin{aligned}
{}_{-\infty}D_x^{\alpha}u(x,t) &= \frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dx^2}\int_{-\infty}^{x}\frac{u(\xi,t)}{(x-\xi)^{\alpha-1}}d\xi,\\
{}_{-\infty}D_x^{\alpha}u(x,t) &= \frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dx^2}\int_{x}^{+\infty}\frac{u(\xi,t)}{(\xi-x)^{\alpha-1}}d\xi.
\end{aligned}
$$

In this work we consider the space fractional KGS equations with high-degree Yukawa interaction — viz.

$$iu_t - (\beta/2)(-\Delta)^{\alpha/2}u + \alpha_1 u\phi + 2\alpha_2|u|^2u\phi = 0, \qquad x\in R, \quad 0<t\le T, \tag{1.2}$$

$$\phi_{tt} - \gamma\Delta\phi + \mu^2\phi - \alpha_1|u|^2 - \alpha_2|u|^4 = 0, \qquad x\in R, \quad 0<t\le T, \tag{1.3}$$

$$u(x,0)=u_0(x), \quad \phi(x,0)=\phi_0(x), \quad \phi_t(x,0)=\phi_1(x), \qquad x\in R, \tag{1.4}$$

where $i^2=-1, 1<\alpha\le 2, u_0, \phi_0, \phi_1$ are fixed smooth functions and $\beta,\gamma,\alpha_1,\alpha_2$ nonnegative parameters.

If $\alpha_2\neq 0$, then (1.2)-(1.4) is a system with a high-degree Yukawa interaction, which consists of Klein-Gordon and space fractional Schrödinger equations. If $\alpha = 2, \beta = 1,$