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Fast Finite Difference Schemes for Time-Fractional Diffusion Equations with a Weak Singularity at Initial Time

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Abstract. A sharp estimate for the $L1$ formula on graded meshes, which approximates the Caputo derivatives of functions with a weak singularity at $t = 0$ is obtained. Combining such approximations with the sum-of-exponential approximations of the kernel, we develop fast difference schemes for one- and two-dimensional fractional diffusion equations, the solutions of which have a weak singularity at the starting time. The proof of the stability and convergence is based on the maximum principle. Numerical examples confirm theoretical estimates.

AMS subject classifications: 65M06, 65M12, 65M15

Key words: Fractional differential equation, difference scheme, fast algorithm, singularity.

1. Introduction

Time fractional diffusion equations find applications in fractional random walks, wave propagation, fluid flows, financial markets [5–7, 14, 19, 28]. However, the non-locality, history dependence and long-range interaction of fractional differential operators lead to high computational cost when approximation methods are used [25]. In this work, we develop fast difference schemes for the time fractional diffusion equation

$$ {}_0^C D_t^\alpha u(x,t) = u_{xx}(x,t) + q(x,t), \quad x \in (0,L), \quad t \in [0,T], \tag{1.1}$$

$$ u(x,0) = \phi(x), \quad x \in [0,L], \tag{1.2}$$

$$ u(0,t) = \varphi_1(t), \quad u(L,t) = \varphi_2(t), \quad t \in (0,T] \tag{1.3}$$

with a weak singularity at the initial time $t = 0$. Note that $q(x,t), \phi(x), \varphi_1(t), \varphi_1(t)$ are known functions and ${}_0^C D_t^\alpha$ is the Caputo derivative of order α — i.e.

$$ {}_0^C D_t^\alpha f(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} \mathrm{d}s, \tag{1.4}$$

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and throughout the paper we always assume that $0 < \alpha < 1$.

The Caputo derivative is often approximated by $L1$ method, which uses uniform meshes and has the accuracy $2-\alpha$ [22,27]. On the other hand, the presence of the kernel $(t-s)^{-\alpha}$ produces solutions of (1.1)-(1.3) with a weak singularity at $t = 0$, so that approximation methods on uniform meshes have a poor convergent rate and high computational cost. Therefore, Yuste and Quintana-Murillo [24, 29] applied $L1$ and $L2$ formulas with non-uniform time step to fractional diffusion and diffusion-wave equations, Mustapha *et al.* [20,21] used the finite difference method with non-uniform time step to a sub-diffusion equation, Zhang *et al.* [31] considered a finite difference scheme for fractional diffusion equations on non-equidistant grids, Li *et al.* [16] employed rectangle and trapezoid formulas in finite difference schemes on non-uniform meshes, Stynes [26] used $L1$ method with graded meshes for fractional diffusion equations, Liao [17] discussed $L1$ formula on nonuniform grids for reaction-subdiffusion equations. These works often provide better error estimates. Nevertheless, in practical applications the computational cost of solving fractional differential equations remains very high. Therefore, recent efforts are directed towards the development of methods reducing storage and computational cost. Thus Ke *et al.* [13] proposed a fast direct method using fast Fourier transform for block triangular Toeplitz-like systems arising in time-fractional partial differential equations, Pang and Sun [23] applied a numerical contour integral method with hyperbolic contour to space-fractional diffusion equations, Wang [10] developed a preconditioned fast Krylov subspace iterative method for space-fractional diffusion equations, Lubich and Schädle [18] used the fast Fourier transform to compute temporal convolutions, Baffet and Hesthaven [2] studied a multipole approximation of the Laplace transform of weakly singular kernels, Zeng *et al.* [30] proposed a unified fast time-stepping method for fractional integral and differentiation operators, Jiang *et al.* [12] used sum-of-exponentials to approximate the kernel $t^{-\alpha-1}$ on the interval $[\delta, T]$. For the approximation of sum-of-exponentials for the kernels, one can also refer to [1,3,9,11,15].

Nonetheless, fast algorithms for fractional differential equations with weakly singular solutions at $t = 0$ are little developed. Here, we use Stynes' graded meshes [4, 26] and Zhang's fast evaluation [12] to construct a fast finite difference method for time-fractional diffusion equations. In particular, in Section 2, we provide a new sharp estimate for Stynes' $L1$ approximation on graded meshes and introduce a fast algorithm based on such meshes. Section 3 deals with a fast $L1$ difference scheme on nonuniform meshes for one dimensional time-fractional diffusion equations. The stability and convergence of the method are analysed in section Section 4, whereas its extension to two dimensional problems is discussed in Section 5. Numerical examples are discussed in Section 6 and our conclusions are in the last section.

2. Preliminary

In this section we present necessary definitions and auxiliary results concerning the approximation of the Caputo derivative.