East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.270318.260518

Vol. **9**, No. 1, pp. 1-12
February 2019

Compound PDE-Based Additive Denoising Solution Combining an Improved Anisotropic Diffusion Model to a 2D Gaussian Filter Kernel

Tudor Barbu[1,*] and C. Moroşanu[2]

[1] *Institute of Computer Science of the Romanian Academy, Iasi, Romania.*
[2] *Department of Mathematics, Al.I. Cuza University of Iasi, Iasi, Romania.*

Received 27 March 2018; Accepted (in revised version) 26 May 2018.

Abstract. A second-order nonlinear anisotropic diffusion-based model for Gaussian additive noise removal is proposed. The method is based on a properly constructed edge-stopping function and provides an efficient detail-preserving denoising. It removes additive noise, overcomes blurring effect, reduces the image staircasing and does not generate multiplicative noise, thus preserving boundaries and all the essential image features very well. The corresponding PDE model is solved by a robust finite-difference based iterative scheme consistent with the diffusion model. The method converges very fast to the model solution, the existence and regularity of which is rigorously proved.

AMS subject classifications: 35Bxx, 94A08, 35K55, 35K60, 35Qxx, 65Nxx

Key words: Image restoration, nonlinear anisotropic diffusion, qualitative properties of solutions, boundary value problems for nonlinear parabolic PDE, Leray-Schauder principle.

1. Introduction

Image denoising is one of fundamental tasks in image processing. However, the classic $2D$ image filters often produce undesired blurring, which affects the edges and other essential image details [10], so that a feature-preserving restoration still represents a serious challenge.

The nonlinear partial differential equations (PDEs) have been increasingly used in the image denoising and restoration in the last three decades. They provide a good solution to the problem since Perona and Malik [18] introduced their celebrated anisotropic diffusion algorithm. Since then, various nonlinear second-order diffusion-based restoration models have been developed — cf. Refs. [5, 23]. On the other hand the total variation (TV) denoising scheme proposed by Rudin *et al.* [19], initiated the development of numerous PDE variational filtering techniques [1, 3, 7, 8, 11, 15, 20, 22].

*Corresponding author. *Email addresses:* `tudor.barbu@iit.academiaromana-is.ro` (T. Barbu), `costica.morosanu@uaic.ro` (C. Moroşanu)

However, although such second-order PDE-based methods remove image blurring and preserve boundaries, they may cause another unintended problem — viz. staircase (or blocky) effect [6]. In contrast, nonlinear fourth-order diffusion-based models inspired by the influential You-Kaveh scheme [24], can successfully remove additive Gaussian noise and overcome the staircase effect — cf. [2,4]. Nevertheless, the over-filtering of the fourth-order diffusion models often affects the image and may produce undesired multiplicative speckle noise.

In this work, we develop a novel PDE-based technique, which successfully removes the additive noise, while avoiding or alleviating all the unintended effects mentioned. It is based on an improved second-order anisotropic diffusion model and a two-dimension Gaussian filter kernel. This model is discussed in Section 2 below. Section 3 deals with a fast-converging numerical approximation scheme based on the finite difference method in [9, 12] and consistent with the model under consideration. A rigorous mathematical analysis of this PDE-based model is provided in Section 4. In particular, we prove the existence and regularity of the classical solution for the corresponding nonlinear second-order diffusion-based scheme. In Section 5, we demonstrate the effectiveness of this restoration approach and compare it with other denoising models using the image quality measures [21]. Our conclusions are in Section 6.

2. A Nonlinear Second-Order Anisotropic Diffusion Model

We consider a novel second-order anisotropic diffusion-based model, which provides an effective detail-preserving image restoration. It is based on a boundary value problem for a nonlinear PDE — viz.

$$\begin{aligned}
&\frac{\partial}{\partial t}u - \eta_u\left(\|\nabla G_\sigma * u\|\right)\nabla\cdot\left(\Psi^u(\|\nabla u\|)\nabla u\right) + \alpha(u-u_0) = 0,\\
&u(0,x_1,x_2) = u_0(x_1,x_2), \quad (x_1,x_2)\in\Omega\subset R^2,\\
&u(t,x_1,x_2) = 0, \quad t\in[0,T], \quad \forall(x_1,x_2)\in\partial\Omega,
\end{aligned} \tag{2.1}$$

where $\alpha \in (0,1]$, u_0 is the observed image, G_σ the Gaussian kernel,

$$G_\sigma(x_1,x_2) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{x_1^2+x_2^2}{2\sigma^2}\right)$$

determined by the standard deviation parameter $\sigma > 0$, and $\|\cdot\|$ refers to the L^2 norm.

The function $\eta_u : (0,\infty) \to (0,\infty)$ in the PDE-based model (2.1) has the form

$$\eta_u(s) = \frac{\left(\lambda s^k + \nu\right)^{1/(k+1)}}{\xi},$$

where $\lambda, \nu \in (0,4]$, $\xi \geq 1.5$, $k \in (0,2]$. It is worth noting that the term $\eta_u\left(\|\nabla G_\sigma * u\|\right)$ controls the speed of this diffusion process and enhances the edges of the corresponding image.

The edge-stopping (diffusivity) function $\Psi^u : [0,\infty) \to [0,\infty)$ in (2.1) has the form

$$\Psi^u(s) = \varepsilon \left(\frac{\delta(u)}{\beta \ln(\delta(u)) + \gamma s_1^2} \right)^{1/3}, \tag{2.2}$$

where $\varepsilon \in (0,2)$, $\gamma \in (1,5]$, $\beta \in (0,1)$, and the conductance parameter δ is defined by

$$\delta(u) := |r\mu(\|\nabla u\|) + \zeta \mathscr{M}(\|\nabla u\|)|, \quad r > 0, \quad \zeta \in (0,1)$$

with the respective averaging and median operators μ and $\mathscr{M}$.

The function (2.2) satisfies the main requirements for a successful restoration [3,23] — e.g. it is positive, monotonically decreasing in $(0,\infty)$ and $\lim_{s\to\infty} \Psi^u(s) = 0$.

Because of the presence of the term $\eta_u(\|\nabla G_\sigma * u\|)$, the nonlinear operator in the Eq. (2.1) does not represent the gradient of the energy functional. Therefore, the proposed second-order nonlinear diffusion-based scheme cannot be obtained from the minimisation of any energy cost functional, so that this scheme is not a variational PDE model.

The restored image is obtained from an observed image by solving the anisotropic diffusion model (2.1). The solution of this equation can be derived by the iterative algorithm introduced in the next section.

3. Consistent Numerical Approximation Algorithm

In this section we propose a robust numerical approximation scheme based on the finite difference method [9,12] for the non-linear model (2.1). More precisely, let h and Δt be, respectively, space and time grids and let

$$x := ih, \quad y := jh, \quad t := n\Delta t \quad \text{for all} \quad i \in \{0,\cdots,I\}, \quad j \in \{0,\cdots,J\}, \quad n \in \{0,\cdots,N\}$$

with the image frame $Ih \times Jh$.

The partial differential equation in (2.1) can be written in the form

$$\frac{\partial}{\partial t}u = \eta_u(\|\nabla G_\sigma * u\|)\left(\frac{\partial}{\partial x_1}(\Psi^u(\|\nabla u\|)u_{x_1}) + \frac{\partial}{\partial x_2}(\Psi^u(\|\nabla u\|)u_{x_2})\right) - \alpha(u - u_0) \tag{3.1}$$

and discretised as follows. First, we compute the terms $\eta_{i,j} = \eta_u(\|(G_\sigma * u)_{i,j}\|)$ and $\Psi_{i,j} = \Psi^u(\|u_{i,j}\|)$, approximating the gradient magnitude by central differences [9,12]:

$$\|u_{i,j}\| = \left(\left(\frac{u_{i+h,j} - u_{i-h,j}}{2h} \right)^2 + \left(\frac{u_{i,j+h} - u_{i,j-h}}{2h} \right)^2 \right)^{1/2}.$$

In addition, the terms

$$\frac{\partial}{\partial x_1}(\Psi^u(\|\nabla u\|)u_{x_1}), \quad \frac{\partial}{\partial x_2}(\Psi^u(\|\nabla u\|)u_{x_2})$$

are, respectively, discretised as

$$\Psi_{i+h/2,j}(u_{i+h,j}-u_{i,j})-\Psi_{i-h/2,j}(u_{i,j}-u_{i-h,j}),$$
$$\Psi_{i,j+h/2}(u_{i,j+h}-u_{i,j})-\Psi_{i,j-h/2}(u_{i,j}-u_{i,j-h}),$$

where

$$\Psi_{i\pm h/2,j}=\frac{\Psi_{i\pm h,j}+\Psi_{i,j}}{2},$$
$$\Psi_{i,j\pm h/2}=\frac{\Psi_{i,j\pm h}+\Psi_{i,j}}{2}.$$

The forward differences are then applied to the time derivative [12] and we arrive at the following implicit discretisation of the Eq. (3.1):

$$\begin{aligned}\frac{u_{i,j}^{n+\Delta t}-u_{i,j}^{n}}{\Delta t}=\eta_{i,j}\Big(&\Psi_{i+h/2,j}(u_{i+h,j}^{n}-u_{i,j}^{n})-\Psi_{i-h/2,j}(u_{i,j}^{n}-u_{i-h,j}^{n})\\ &+\Psi_{i,j+h/2}(u_{i,j+h}^{n}-u_{i,j}^{n})-\Psi_{i,j-h/2}(u_{i,j}^{n}-u_{i,j-h}^{n})\Big)-\alpha(u_{i,j}^{n}-u_{i,j}^{0}).\end{aligned}\tag{3.2}$$

Using the parameters $\Delta t=1$ and $h=1$, we rewrite the implicit approximation algorithm (3.2) as an explicit numerical approximation scheme — viz.

$$\begin{aligned}u_{i,j}^{n+1}=&\,u_{i,j}^{n}\left[1-\alpha-\eta_{i,j}\left(\Psi_{i+1/2,j}+\Psi_{i-1/2,j}+\Psi_{i,j+1/2}+\Psi_{i,j-1/2}\right)\right]\\ &+u_{i+1,j}^{n}\eta_{i,j}\Psi_{i+1/2,j}+u_{i-1,j}^{n}\eta_{i,j}\Psi_{i-1/2,j}\\ &+u_{i,j+1}^{n}\eta_{i,j}\Psi_{i,j+1/2}+u_{i,j-1}^{n}\eta_{i,j}\Psi_{i,j-1/2}+\alpha u_{i,j}^{0},\end{aligned}\tag{3.3}$$

where $u_{0,j}^{n}=u_{1,j}^{n}$, $u_{I,j}^{n}=u_{I+1,j}^{n}$, $u_{i,0}^{n}=u_{i,1}^{n}$ and $u_{i,J}^{n}=u_{i,J+1}^{n}$. The iterative numerical approximation algorithm (3.3) is stable, consistent with nonlinear diffusion-based model (2.1) and converges fast to the exact solution representing the restored image.

4. The Validity of the Diffusion Model

In this section we study the validity of the above nonlinear model. In particular, we analyse the solvability of the model and the regularity of its solutions.

Let k be a positive integer and $1\le p\le\infty$. We denote by $W_p^{k,2k}(Q)$ the Sobolev space

$$W_p^{k,2k}(Q):=\left\{y\in L^p(Q):\frac{\partial^r}{\partial t^r}\frac{\partial^q}{\partial x^q}\,y\in L^p(Q),\text{for }2r+q\le k\right\},$$

— cf. Ref. [13, p. 5], where $Q:=(0,T]\times\Omega$. Moreover, let $C^{1,2}(\bar{Q})$ $(C^{1,2}(Q))$ denote the set of all functions continuous in $\bar{Q}$ (Q) along with their derivatives u_t, u_x, u_{xx} and $W_\infty^{2-2/p}(\Omega)$, $W_p^{l,l/2}(\Sigma)$ the corresponding Sobolev spaces with a non integral l — cf. [13, pp. 70, 81].

Now we consider the problem (2.1) in a bounded domain $\Omega \subset \mathbb{R}^2$ with the boundary $\partial\Omega \in C^2$ for a finite time $T > 0$, so that

$$\begin{aligned} &\frac{\partial}{\partial t}u(t,x_1,x_2) = \eta_u\left(\|\nabla G_\sigma * u\|\right) \operatorname{div}\left(\Psi^u(\|\nabla u\|)\nabla u\right) \\ &\quad -\alpha(u(t,x_1,x_2)-u_0(x_1,x_2)) + f(t,x_1,x_2) \quad \text{in } Q, \\ &\frac{\partial}{\partial \nu}u(t,x_1,x_2) = 0 \quad \text{on } \Sigma, \\ &u(0,x_1,x_2) = u_0(x_1,x_2) \quad \text{on } \Omega, \end{aligned} \tag{4.1}$$

where $\Sigma := (0,T] \times \partial\Omega$, $u_0(x_1,x_2) \in W_\infty^{2-2/p}(\Omega)$, $p \geq 2$ and $\partial u_0(x_1,x_2)/\partial\nu = 0$. The other terms in (4.1) are as before.

Definition 4.1. Any solution of the problem (4.1) is called the classical solution if it is continuous in $\bar{Q}$ and has continuous derivatives u_t, u_x, u_{xx} in Q.

For the sake of convenience, we rewrite the problem (4.1) in the following equivalent form

$$\begin{aligned} &\frac{\partial}{\partial t}u(t,x_1,x_2) - \eta_u\left(\|\nabla G_\sigma * u\|\right) \frac{\partial}{\partial u_{x_j}}\left(\Psi^u(\|\nabla u\|)u_{x_i}\right)u_{x_j x_i} \\ &\quad + A(t,x_1,x_2,u,u_{x_i}) = \alpha\, u_0(x_1,x_2) + f(t,x_1,x_2) \quad \text{in } Q, \\ &\frac{\partial}{\partial \nu}u(t,x_1,x_2) = 0 \quad \text{on } \Sigma, \\ &u(0,x_1,x_2) = u_0(x_1,x_2) \quad \text{on } \Omega, \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} A(t,x_1,x_2,u,u_{x_i}) = -\eta_u\left(\|\nabla G_\sigma * u\|\right)\left[\frac{\partial}{\partial u}\left(\Psi^u(\|\nabla u\|)u_{x_i}\right)u_{x_i} + \frac{\partial}{\partial x_i}\left(\Psi^u(\|\nabla u\|)u_{x_i}\right)\right] \\ + \alpha\, u(t,x_1,x_2) \end{aligned}$$

and

$$u_{x_i} := \frac{\partial}{\partial x_i}u(t,x_1,x_2), \quad u_{x_j x_i} := \frac{\partial^2}{\partial x_j \partial x_i}u(t,x_1,x_2), \quad i,j = 1,2.$$

Our study of the solvability of the problem (4.2) in the space $W_p^{1,2}(Q)$ is based on the Leray-Schauder degree theory and the L^p-theory of linear and quasi-linear parabolic equations [13]. More precisely, the following theorem is true.

Theorem 4.1. *Let $u(t,x_1,x_2) \in C^{1,2}(Q)$ be a classical solution of Eq.* (4.2) *such that*

$$\frac{\partial}{\partial \nu}u(t,x_1,x_2) = 0$$

on the lateral surface Σ of the cylinder Q and for positive numbers M, M_1, M_2 and M_3 one has

I_1. *If $(t,x_1,x_2)\in Q$, then $|u(t,x_1,x_2)|<M$ and for arbitrary q, the function $\Psi^u(\|\nabla u\|)q$ is continuous, differentiable with respect to $x=(x_1,x_2)$, u and q and satisfies the condition of uniform parabolicity — i.e.*

$$\nu|y|^2 \le \frac{\partial}{\partial q_j}(\Psi^u(\|\nabla u\|)q)y_i y_j \le \mu|y|^2, \quad \nu>0, \quad y\in\mathbb{R}^n,$$

and

$$\begin{aligned}&\left[|\Psi^u(\|\nabla u\|)u_{x_i}|+\left|\frac{\partial}{\partial u}(\Psi^u(\|\nabla u\|)u_{x_i})\right|\right](1+|q|)\\ &\quad+\left|\frac{\partial}{\partial x_1}(\Psi^u(\|\nabla u\|)u_{x_1})\right|+\left|\frac{\partial}{\partial x_2}(\Psi^u(\|\nabla u\|)u_{x_1})\right|\\ &\quad+\left|\frac{\partial}{\partial x_1}(\Psi^u(\|\nabla u\|)u_{x_2})\right|+\left|\frac{\partial}{\partial x_2}(\Psi^u(\|\nabla u\|))u_{x_2}\right|\\ &\quad+|u(t,x_1,x_2)|\le\mu(1+|q|)^2.\end{aligned} \tag{4.3}$$

I_2. *For any sufficiently small $\varepsilon>0$, the functions η_u, u and Ψ^u satisfy the inequalities*

$$\eta_u(\|\nabla G_\sigma * u\|)<M_1, \quad \|u\|_{L^s(Q)}\le M_3, \quad \|\Psi^u(\|\nabla u\|)u_{x_i}\|_{L^r(Q)}<M_2, \quad i=1,2,$$

where

$$r=\begin{cases}\max\{p,4\}, & p\neq 4,\\ 4+\varepsilon, & p=4,\end{cases} \quad \textit{and} \quad s=\begin{cases}\max\{p,2\}, & p\neq 2,\\ 2+\varepsilon, & p=2.\end{cases}$$

Then, for any $f\in L^p(Q)$ and $u_0\in W_\infty^{2-2/p}(\Omega)$, $p\neq 3/2$, the problem (4.2) has a solution $u\in W_p^{1,2}(Q)$, such that

$$\|u\|_{W_p^{1,2}(Q)}\le C\left(\|u_0\|_{W_\infty^{2-2/p}(\Omega)}+\|f\|_{L^p(Q)}\right), \tag{4.4}$$

where the constant $C>0$ is independent of u and f.

Proof. To prove this theorem, we use the Leray-Schauder principle. Consider the Banach space $B=W_p^{0,1}(Q)$ endowed with the norm $\|v\|_B=\|v\|_{L^p(Q)}+\|v_x\|_{L^p(Q)}$, and a nonlinear operator H defined by

$$H(v,\lambda):=u(v,\lambda) \quad \text{for all} \quad (v,\lambda)\in W_p^{0,1}(Q)\times[0,1], \tag{4.5}$$

where $u(v,\lambda)$ is the unique solution to the following linear boundary value problem

$$\begin{aligned}&\frac{\partial}{\partial t}u(t,x_1,x_2)-\left[\lambda\eta_v(\|\nabla G_\sigma * v\|)\frac{\partial}{\partial v_{x_j}}(\Psi^v(\|\nabla v\|)v_{x_i})+(1-\lambda)\delta_i^j\right]\\ &\quad\times u_{x_ix_j}=-\lambda\left[A(t,x_1,x_2,v,v_{x_i})-\alpha\, u_0(x_1,x_2)+f(t,x_1,x_2)\right] \quad \text{in } Q,\\ &\frac{\partial}{\partial \nu}u(t,x_1,x_2)=0 \quad \text{on } \Sigma,\\ &u(0,x_1,x_2)=\lambda u_0(x_1,x_2) \quad \text{on } \Omega.\end{aligned} \tag{4.6}$$

Our theorem will be proved if we show that the nonlinear operator H has two properties:

A. The operator H is well-defined.

B. The operator H is continuous and compact.

We start with the definition of the operator H.

A. The operator H is well-defined if the problem (4.6) has a unique solution. The Eq. (4.6) shows that if $v \in W_p^{0,1}(Q)$, then $A(t,x_1,x_2,v,v_{x_i}) + f(t,x_1,x_2) \in L^p(Q)$ and according to [13, p. 341-342], the problem (4.6) has unique solution u such that

$$u = u(v,\lambda) \in W_p^{1,2}(Q), \quad \forall v \in W_p^{0,1}(Q), \quad \forall \lambda \in [0,1].$$

Taking into account that $W_p^{1,2}(Q) \subset W_p^{0,1}(Q)$ — cf. [14, p. 24], we obtain that $H(v,\lambda) = u \in W_p^{0,1}(Q)$ for all $v \in W_p^{0,1}(Q)$ and $\lambda \in [0,1]$.

B. Let us now show that H is continuous and compact. Let $v^n \to v$ in $W_p^{0,1}(Q)$ and $\lambda_n \to \lambda$ in $[0,1]$. Using the notation

$$u^{n,\lambda_n} = H(v^n,\lambda_n), \quad u^{n,\lambda} = H(v^n,\lambda) \quad \text{and} \quad u^{\lambda} = H(v,\lambda)$$

and considering the difference $H(v^n,\lambda_n) - H(v^n,\lambda)$, we obtain from the Eqs. (4.5) and (4.6) that

$$\begin{aligned}
&\frac{\partial}{\partial t}(u^{n,\lambda_n} - u^{n,\lambda}) - \Big[\lambda \eta_{v^n}(\|\nabla G_\sigma * v^n\|)\frac{\partial}{\partial v^n_{x_j}}(\Psi^{v^n}(\|\nabla v^n\|)v^n_{x_i}) + (1-\lambda)\delta_i^j\Big] \\
&\quad \times (u^{n,\lambda_n}_{x_i x_j} - u^{n,\lambda}_{x_i x_j}) = -(\lambda_n - \lambda)\Big\{\Big[\eta_{v^n}(\|\nabla G_\sigma * v^n\|)\frac{\partial}{\partial v^n_{x_j}}(\Psi^{v^n}(\|\nabla v^n\|)v^n_{x_i}) - \delta_i^j\Big]u^{n,\lambda_n}_{x_i x_j} \\
&\quad + A(t,x,v^n,v^n_{x_i}) - \alpha\, u_0(x_1,x_2) + f(t,x_1,x_2)\Big\} \quad \text{in } Q, \\
&\frac{\partial}{\partial \nu}(u^{n,\lambda_n} - u^{n,\lambda}) = 0 \quad \text{on } \Sigma, \\
&(u^{n,\lambda_n} - u^{n,\lambda})(0,x_1,x_2) = (\lambda_n - \lambda)u_0(x_1,x_2) \quad \text{in } \Omega.
\end{aligned} \tag{4.7}$$

The right-hand side in (4.7) belongs to $L^p(Q)$, since $u^{n,\lambda_n} \in W_p^{1,2}(Q)$. Therefore, the L^p-theory of PDE yields the estimate

$$\begin{aligned}
\|u^{n,\lambda_n} - u^{n,\lambda}\|_{W_p^{1,2}(Q)} \le C|\lambda_n - \lambda| \Bigg(\Bigg\| \Big(\eta_{v^n}(\|\nabla G_\sigma * v^n\|)\frac{\partial}{\partial v^n_{x_j}}(\Psi^{v^n}(\|\nabla v^n\|)v^n_{x_i}) - \delta_i^j\Big)u^{n,\lambda_n}_{x_i x_j} \Bigg\|_{L^p(Q)} \\
+ \|A(t,x,v^n,v^n_{x_i})\|_{L^p(Q)} + \|u_0\|_{W_\infty^{2-2/p}(\Omega)} + \|f\|_{L^p(Q)} \Bigg)
\end{aligned}$$

with a constant $C(|\Omega|, p, \alpha, M, M_1, M_2, M_3)$.

The inequality (4.3), condition I_2 and the inclusion $u^{n,\lambda_n}_{x_i x_j} \in L^p(Q)$ imply the boundedness of $A(t,x,v^n,v^n_{x_i})$, $\Big(\eta_{v^n}(\|\nabla G_\sigma * v^n\|)\frac{\partial}{\partial v^n_{x_j}}(\Psi^{v^n}(\|\nabla v^n\|)v^n_{x_i}) - \delta_i^j\Big)u^{n,\lambda_n}_{x_i x_j}$, u_0 and f in $L^p(Q)$, and since $\lambda_n \to \lambda$, we obtain

$$\|u^{n,\lambda_n} - u^{n,\lambda}\|_{W_p^{1,2}(Q)} \to 0 \quad \text{for} \quad n \to \infty. \tag{4.8}$$

In order to evaluate the difference $H(v^n,\lambda)-H(v,\lambda)$, we use (4.5) and (4.6), so that

$$
\begin{aligned}
&\frac{\partial}{\partial t}(u^{n,\lambda}-u^{\lambda})-\left[\lambda\eta_{v^n}(\|\nabla G_\sigma * v^n\|)\frac{\partial}{\partial v^n_{x_j}}(\Psi^{v^n}(\|\nabla v^n\|)v^n_{x_i})+(1-\lambda)\delta_i^j\right]\\
&\quad\times(u^{n,\lambda}_{x_\lambda x_j}-u^{\lambda}_{x_i x_j})=-\lambda\left[\eta_{v^n}(\|\nabla G_\sigma * v^n\|)\frac{\partial}{\partial v^n_{x_j}}(\Psi^{v^n}(\|\nabla v^n\|)v^n_{x_i})\right.\\
&\quad\left.-\eta_v(\|\nabla G_\sigma * v\|)\frac{\partial}{\partial v_{x_j}}(\Psi^{v}(\|\nabla v\|)v_{x_i})\right]u^{\lambda}_{x_i x^j}\\
&\quad-\lambda[A(t,x,v^n,v^n_{x_i}-A(t,x,v,v_{x_i})]\quad\text{in } Q,\\
&\frac{\partial}{\partial \nu}(u^{n,\lambda}-u^{\lambda})=0\quad\text{on }\Sigma,\\
&(u^{n,\lambda}-u^{\lambda})(0,x_1,x_2)=0\quad\text{on }\Omega.
\end{aligned}
\tag{4.9}
$$

Using the L^p-theory of PDE again, we arrive at the estimate

$$
\begin{aligned}
\|u^{n,\lambda}-u^{\lambda}\|_{W^{1,2}_p(Q)}\le C\Bigg[\Bigg\|\Big(&\eta_{v^n}(\|\nabla G_\sigma * v^n\|)\frac{\partial}{\partial v^n_{x_j}}(\Psi^{v^n}(\|\nabla v^n\|)v^n_{x_i})\\
&-\eta_v(\|\nabla G_\sigma * v\|)\frac{\partial}{\partial v_{x_j}}(\Psi^{v}(\|\nabla v\|)v_{x_i})\Big)u^{\lambda}_{x_i x_j}\Bigg\|_{L^p(Q)}\\
&+\|A(t,x,v^n,v^n_{x_i})-A(t,x,v,v_{x_i})\|_{L^p(Q)}\Bigg]
\end{aligned}
$$

with a constant C. Since all terms in the right-hand side of this inequality are bounded and v^n converges to v in $W^{0,1}_p(Q)$, it follows that

$$
\|u^{n,\lambda}-u^{\lambda}\|_{W^{1,2}_p(Q)}\to 0\quad\text{as}\quad n\to\infty. \tag{4.10}
$$

Making use of the relations (4.8) and (4.10), we show the continuity of the nonlinear operator H. Moreover, the mapping H defined by (4.5) is compact, what can easily be seen by writing it as the composition

$$
B=W^{0,1}_p(Q)\times[0,1]\to W^{1,2}_p(Q)\hookrightarrow B=W^{0,1}_p(Q),
$$

where the second map is an compact inclusion due to Lions-Peeter embedding theorem — cf. Ref. [14, p. 21].

Now we establish the existence of a number $\delta>0$ such that

$$
(u,\lambda)\in W^{0,1}_p(Q)\times[0,1]\quad\text{with}\quad u=H(u,\lambda)\Rightarrow\|u\|_B<\delta. \tag{4.11}
$$

The equality $u = H(u,\lambda)$ in (4.11) is equivalent to

$$\begin{aligned}
&\frac{\partial}{\partial t}u(t,x_1,x_2) - \Big[\lambda\eta_u(\|\nabla G_\sigma * u\|)\frac{\partial}{\partial u_{x_j}}(\Psi^u(\|\nabla u\|)u_{x_i}) + (1-\lambda)\delta_i^j\Big] \\
&\quad\times u_{x_i x_j} = -\lambda\Big[A(t,x_1,x_2,u,u_{x_i}) - \alpha\, u_0(x_1,x_2) + f(t,x_1,x_2)\Big] \quad \text{in } Q, \\
&\frac{\partial}{\partial \nu}u(t,x_1,x_2) = 0 \quad \text{on } \Sigma, \\
&u(0,x_1,x_2) = \lambda u_0(x_1,x_2) \quad \text{on } \Omega.
\end{aligned} \tag{4.12}$$

Taking into account the L^p-theory of PDFs, assumptions (4.3) and I_2, for $p \neq 3/2$ we conclude that

$$\|u\|_{W_p^{1,2}(Q)} \leq C\left(\|u_0\|_{W_\infty^{2-2/p}(\Omega)} + \|f\|_{L^p(Q)}\right) \tag{4.13}$$

with a constant $C(|\Omega|, p, \alpha, \nu, M, M_1, M_2, M_3) > 0$. This inequality and the embedding $W_p^{1,2}(Q) \subset W_p^{0,1}(Q)$ yield

$$\|u\|_{W_p^{0,1}(Q)} \leq C\|u\|_{W_p^{1,2}(Q)},$$

thus confirming the validity of (4.11).

Considering the ball

$$B_\delta := \{u \in B : \|u\|_B < \delta\}$$

we note that the inequality (4.11) implies that $H(u,\lambda) \neq u$ for any $u \in \partial B_\delta$ and $\lambda \in [0,1]$, provided that $\delta > 0$ is sufficiently large. Moreover, following the arguments of [16] and [17], we conclude that the problem (4.2) has a solution $u \in W_p^{1,2}(Q)$, and the inequality (4.13) leads to the estimate (4.4). □

Remark 4.1. The nonlinear operator H in (4.5) depends on $\lambda \in [0,1]$ and for $\lambda = 1$ its fixed points are the solutions of (4.2)

5. Restoration Experiments and Method Comparison

The proposed diffusion-based restoration approach has been successfully tested on the images corrupted by white additive Gaussian noise. The images were acquired from the USC - SIPI database.

The performance of this technique has been assessed by the Peak Signal to Noise Ratio (PSNR), the Signal to Noise Ratio (SNR) and Mean-squared Error (MSE) — cf. Ref. [17]. The following set of the model parameters

$$\begin{aligned}
&\alpha = 0.6, \quad \lambda = 1.3, \quad \nu = 1.5, \quad k = 0.35, \quad \xi = 1.66, \quad \varepsilon = 0.5, \\
&\gamma = 3.7, \quad \beta = 0.4, \quad r = 0.6, \quad \zeta = 0.25, \quad N = 12
\end{aligned}$$

ensures optimal image restoration. It was empirically identified by the trial and error method. It is worth noting that the method successfully removes additive noise, overcomes blurring effect, reduces the image staircasing and does not generate multiplicative noise,

Table 1: Average PSNR and SSIM for several filtering models.

Image Filtering technique	Peak Signal to Noise Ratio	Structural Similarity Index
The proposed AD restoration	29.6128 (dB)	0.8751
Gaussian 2D filter	23.1857 (dB)	0.4532
Average filter	24.9836 (dB)	0.5174
Wiener 2D filter	25.7205 (dB)	0.8271
Perona-Malik 1	25.4753 (dB)	0.6789
Perona-Malik 2	26.1462 (dB)	0.7514
ROF - TV Denoising	27.8543 (dB)	0.8356

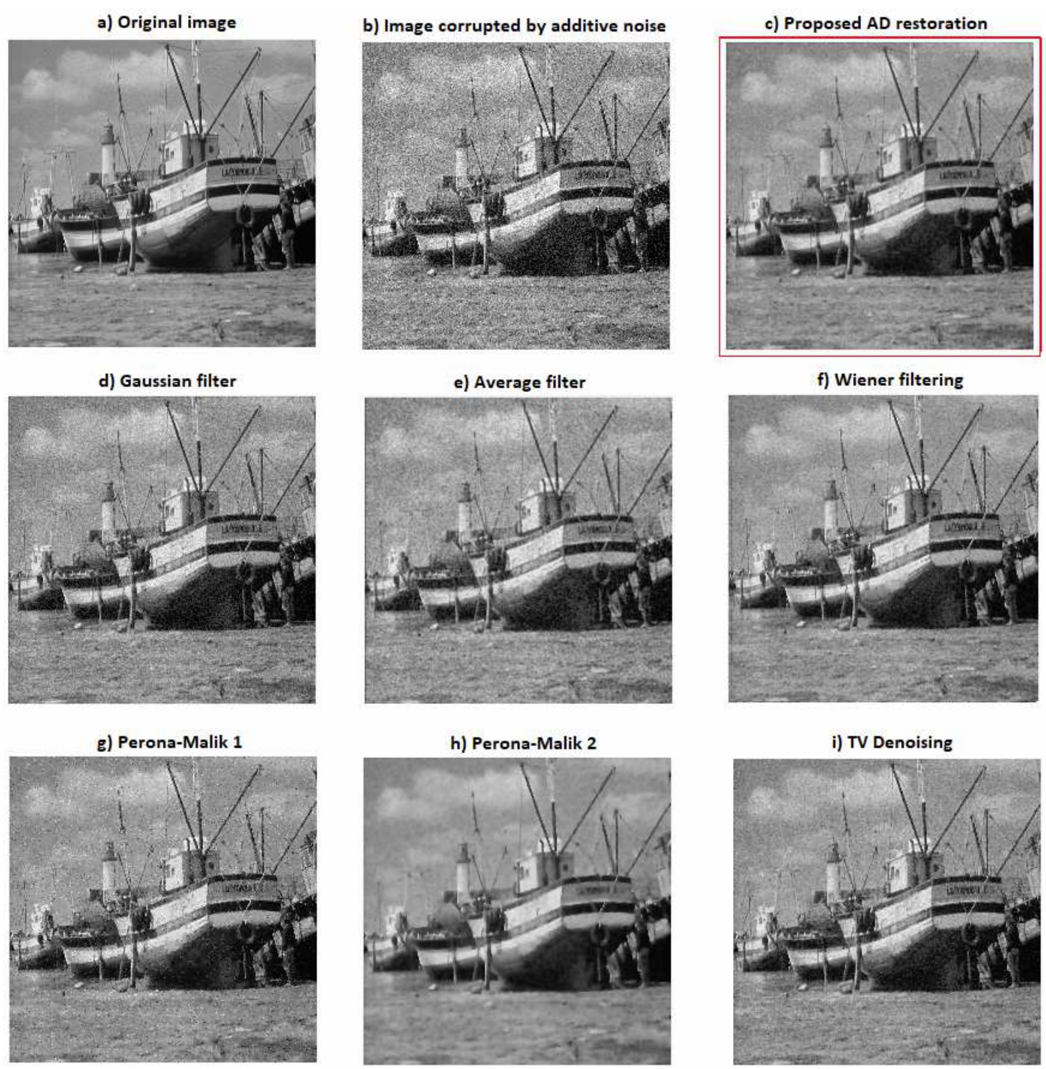

Figure 1: The restoration of the Boat image.

thus preserving boundaries and all the essential image features very well. Moreover, since the iterative numerical algorithm for the corresponding PDE converges fast to the optimal restoration, the method requires a low execution time, which depends on the amount of noise present.

Comparing this method with the existing PDE and non-PDE restoration models, we observe that it clearly outperforms the conventional two-dimension image filters [10] such as average, Gaussian 2D and Wiener, and also the linear PDE-based filters, providing a better denoising and avoiding the undesired blurring effect. It also outperforms the non-linear second-order diffusion-based and variational schemes motivated by Perona-Malik and ROF-TV denoising models [5,19,23], which encounter difficulties with staircase effect. Moreover, the solutions obtained by this method are better than even those found by the fourth-order You-Kaveh alike PDE models [24], since there is no multiplicative (speckle) noise, the edgesand, the other image details are better preserved and the execution is much faster.

Table 1 shows the average PSNR and SSIM (Structural Similarity Index) for various denoising methods, with the introduced anisotropic diffusion (AD) approach achieving the best results. Note that the other methods data are taken of [21].

Fig. 1 demonstrates one of numerous filtering simulations for the Boat image. The original image from USC-SIPI database, is corrupted by the Gaussian additive noise characterised by $\mu = 0.11$ and the variance 0.02 — cf. Fig. 1(b). The restoration results for various methods are presented in Figs. 1(c)-1(i).

6. Conclusions

We propose a novel second-order nonlinear anisotropic diffusion-based model for Gaussian additive noise removal. The method is based on a properly constructed edge-stopping function and provides an efficient detail-preserving denoising. The corresponding PDE model is solved by a robust finite-difference based iterative approximation scheme consistent with the diffusion model. The method converges very fast to the model solution, the existence and regularity of which is rigorously proved.

The proposed restoration method successfully removes additive noise, overcomes blurring effect, reduces the image staircasing and does not generate multiplicative noise, thus preserving boundaries and all the essential image features very well. It outperforms many other approaches and can be used in edge detection and object detection models and in image inpainting problems.

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