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Lump and Interaction Solutions of Linear PDEs in $(3+1)$-Dimensions

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Abstract. Linear partial differential equations in $(3+1)$-dimensions consisting of all mixed second-order derivatives are considered, and Maple symbolic computations are made to construct their lump and interaction solutions, including lump-periodic, lump-kink and lump-soliton solutions.

AMS subject classifications: 35Q51, 35Q53, 37K40

Key words: Symbolic computation, lump solution, interaction solution.

1. Introduction

Lump solutions are special exact solutions of partial differential equations (PDFs), which describe important wave phenomena [1,29]. Specific lumps can be obtained from solitons through taking long wave limits [30]. Other classes of solutions to integrable equations include positons and complxitons [16,35], and interaction solutions [26], which exhibit more diverse nonlinear wave phenomena.

From a mathematical point of view, soliton solutions are exponentially localised in time and in all space directions, whereas lump solutions are rationally localised in all space

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directions. Let P be a polynomial, and D_x and D_t be the Hirota bilinear derivatives. Based on the Hirota bilinear form

$$P(D_x, D_t)f \cdot f = 0,$$

the corresponding N-soliton solution in $(1+1)$-dimensions can take the form

$$f = \sum_{i,j=1}^{N} \exp\Big(\sum_{i=1}^{N} \mu_i \xi_i + \sum_{i<j} \mu_i \mu_j a_{ij}\Big),$$

where $\mu_j \in \{0,1\}$, $j = 1, 2, \cdots, N$, and

$$\begin{aligned} \xi_i &= k_i x - \omega_i t + \xi_{i,0}, \quad 1 \le i \le N, \\ \mathrm{e}^{a_{ij}} &= -\frac{P(k_i - k_j, \omega_j - \omega_i)}{P(k_i + k_j, \omega_j + \omega_i)}, \quad 1 \le i < j \le N, \end{aligned}$$

with the wave numbers k_i and the wave frequencies ω_i satisfying the dispersion relation, and $\xi_{i,0}$ being arbitrary shifts.

It is known [21] that the KPI equation

$$(u_t + 6uu_x + u_{xxx})_x - u_{yy} = 0$$

has the lump solution

$$u = 2(\ln f)_{xx}, \quad f = \big(a_1 x + a_2 y + a_3 t + a_4\big)^2 + \big(a_5 x + a_6 y + a_7 t + a_8\big)^2 + a_9,$$

where

$$a_3 = \frac{a_1 a_2{}^2 - a_1 a_6{}^2 + 2, a_2 a_5 a_6}{a_1{}^2 + a_5{}^2}, \quad a_7 = \frac{2a_1 a_2 a_6 - a_2{}^2 a_5 + a_5 a_6{}^2}{a_1{}^2 + a_5{}^2}, \quad a_9 = \frac{3(a_1{}^2 + a_5{}^2)^3}{(a_1 a_6 - a_2 a_5)^2},$$

and $a_1 a_6 - a_2 a_5 \neq 0$. The last condition guarantees the rational localisation in all directions in the (x, y)-plane. There are many other integrable equations with lump solutions — e.g. three-dimensional three-wave resonant interaction [8], BKP equation [5, 38], Davey-Stewartson equation II [30], Ishimori-I equation [7] — cf. also Refs. [27, 46]. Moreover, non-integrable equations can also have lump solutions [2, 24, 43, 44], and there are interaction solutions of nonlinear integrable equation in (2+1)-dimensions, including lump-soliton interaction solutions [25, 39, 41, 42] and lump-kink interaction solutions [9, 31, 45, 48]. In $(3+1)$-dimensions, only the integrable Jimbo-Miwa equation has been known to have lump-type solutions, rationally localised in almost all (but not all) space directions. On the other hand, all analytical rational solutions of the $(3+1)$-dimensional Jimbo-Miwa equation in [22, 40, 47] and of the $(3+1)$-dimensional Jimbo-Miwa like equation in [6] are not rationally localised in all space directions, either. Therefore, in $(3+1)$-dimensions, lump and interaction solutions of PDEs are interesting objects to study.

The aims of this work is to show the existence of lump and interaction solutions of PDEs in $(3+1)$-dimensions. A class of particular examples of equations in $(3+1)$-dimensions is considered. In particular, we provide explicit representations of lump, mixed lump-periodic and mixed lump-soliton solutions of a class of (3+1)-dimensional linear PDEs. Using Maple symbolic computations, we establish sufficient conditions for the existence of lumps, and present lump and interaction solutions of the equations under consideration. Concluding remarks are given in the last section.

2. Diverse Lump and Interaction Solutions

Let $u = u(x, y, z, t)$ be a real function of real variables x, y, z and t. We consider the following class of linear PDEs, consisting of all mixed second-order derivative terms:

$$\alpha_1 u_{xy} + \alpha_2 u_{xz} + \alpha_3 u_{xt} + \alpha_4 u_{yz} + \alpha_5 u_{yt} + \alpha_6 u_{zt} = 0, \tag{2.1}$$

where α_i, $i = 1, 2, \cdots, 6$ are real constant coefficients and subscripts denote partial differentiation.

Real-valued solutions of (2.1) are sought in the form

$$u = v(\xi_1, \xi_2, \xi_3, \xi_4), \tag{2.2}$$

where

$$\xi_i = a_i x + b_i y + c_i z + d_i t + e_i, \quad i = 1, 2, 3, 4$$

and a_i, b_i, c_i, d_i and e_i are real constants to be determined. Substituting (2.2) in (2.1), we obtain

$$\sum_{i=1}^{4}\sum_{j=i}^{4} w_{ij} v_{\xi_i \xi_j} = 0,$$

where w_{ij}, $i, j = 1, 2, 3, 4$ are quadratic functions of a_i, b_i, c_i and d_i. Setting $w_{ij} = 0$ for all present combinations of i and j, we arrive at the system of equations

$$\begin{aligned}
&\alpha_1 a_i b_i + \alpha_2 a_i c_i + \alpha_3 a_i d_i + \alpha_4 b_i c_i + \alpha_5 b_i d_i + \alpha_6 c_i d_i = 0, \quad 1 \le i \le 4, \\
&\alpha_1(a_i b_j + a_j b_i) + \alpha_2(a_i c_j + a_j c_i) + \alpha_3(a_i d_j + a_j d_i) \\
&\quad + \alpha_4(b_i c_j + b_j c_i) + \alpha_5(b_i d_j + b_j d_i) + \alpha_6(c_i d_j + c_j d_i) = 0, \quad 1 \le i < j \le 4.
\end{aligned}$$

Various solutions of this system of quadratic equations can be derived by Maple symbolic computations, but we chose only two interesting sets of solutions — viz.

$$\begin{aligned}
&\Big\{ b_1 = c_1 = c_2 = 0, \quad d_2 = \frac{a_2 d_1}{a_1}, \quad d_3 = \frac{a_3 d_1}{a_1}, \quad d_4 = \frac{a_4 d_1}{a_1}, \\
&\alpha_1 = -\frac{d_1}{a_1}\alpha_5, \quad \alpha_2 = -\frac{d_1}{a_1}\alpha_6, \quad \alpha_3 = \alpha_4 = 0 \Big\},
\end{aligned}$$

and

$$\begin{aligned}
&\Big\{ b_1 = c_1 = 0, \quad a_3 = \frac{a_2 c_3 d_1 - a_1 c_3 d_2 + a_1 c_2 d_3}{c_2 d_1}, \quad a_4 = \frac{a_2 c_4 d_1 - a_1 c_4 d_2 + a_1 c_2 d_4}{c_2 d_1}, \\
&\alpha_1 = \frac{a_2 d_1 - a_1 d_2}{a_1 c_2}\alpha_5, \quad \alpha_2 = \alpha_3 = 0, \quad \alpha_4 = -\frac{d_1}{a_1}\alpha_5, \quad \alpha_6 = 0 \Big\}.
\end{aligned}$$

The parameters not determined above, are arbitrary provided that the resulting formulas are well defined. Although the parameters in these sets generate lumps and the corresponding interaction solutions, they all satisfy the determinant equation

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0,$$

which implies that the resulting solutions are not rogue waves.

Taking into account the two solutions above, we consider two types of equations.

Case 1: Setting $a_1 = -d_1$ leads to the following reduced linear PDE:

$$u_{xy} + u_{xz} + u_{ty} + u_{tz} = 0, \tag{2.3}$$

which has the solutions of the form

$$u = (\ln f)_{xx}, \quad f = \xi_1^{2n_1} + \xi_2^{2n_2} + \xi_3^{2n_3} + g(\xi_4),$$

with arbitrary natural numbers n_i, $i = 1, 2, 3$, an arbitrary function g and the wave variables

$$\begin{aligned} \xi_1 &= a_1 x - a_1 t + e_1, \\ \xi_2 &= a_2 x + b_2 y - a_2 t + e_2, \\ \xi_3 &= a_3 x + b_3 y + c_3 z - a_3 t + e_3, \\ \xi_4 &= a_4 x + b_4 y + c_4 z - a_4 t + e_4. \end{aligned}$$

Therefore, if $g(\xi_4)$ is one of the functions

$$\beta_1, \quad \beta_2 + \beta_3 \cos \xi_4, \quad \beta_4 e^{\xi_4}, \quad \beta_5 \cosh \xi_4,$$

with constants β_i such that f takes only positive values, then we obtain lump solutions and also interaction solutions of the Eq. (2.3) such as lump-periodic, lump-kink and lump-soliton solutions. For example, if $n_1 = n_2 = n_3 = 1$, then

$$\begin{aligned} u = \frac{f_{xx} f - f_x^2}{f^2} &= \frac{2a_1^2 + 2a_2^2 + 2a_3^2 + a_4^2 g''(\xi_4)}{f} \\ &\quad - \frac{[2a_1\xi_1 + 2a_2\xi_2 + 2a_3\xi_3 + a_4 g'(\xi_4)]^2}{f^2}. \end{aligned} \tag{2.4}$$

Case 2: Setting $a_1 = -d_1,\ a_2 = -2d_2$ and $c_2 = d_2$, leads to another reduced linear PDE — viz.

$$u_{xy} + u_{yz} + u_{ty} = 0, \tag{2.5}$$

which has the solutions of the form

$$u = (\ln f)_{xx}, \quad f = \xi_1^{2n_1} + \xi_2^{2n_2} + \xi_3^{2n_3} + g(\xi_4),$$

with arbitrary natural numbers n_i, $i = 1, 2, 3$, an arbitrary function g and the wave variables

$$\begin{aligned}
\xi_1 &= a_1 x - a_1 t + e_1, \\
\xi_2 &= -2c_2 x + b_2 y + c_2 z + c_2 t + e_2, \\
\xi_3 &= -(c_3 + d_3)x + b_3 y + c_3 z + d_3 t + e_3, \\
\xi_4 &= -(c_4 + d_4)x + b_4 y + c_4 z + d_4 t + e_4.
\end{aligned}$$

Therefore, if $g(\xi_4)$ is one of the functions

$$\beta_1, \quad \beta_2 + \beta_3 \sin \xi_4, \quad \beta_4 \cosh \xi_4,$$

with constants β_i such that f takes only positive values, then we obtain lump solutions and also interaction solutions of the Eq. (2.5) such as lump-periodic and lump-soliton solutions. For example, if $n_1 = n_2 = n_3 = 1$, then

$$\begin{aligned}
u = \frac{f_{xx} f - f_x^2}{f^2} &= \frac{2a_1^2 + 8c_2^2 + 2(c_3 + d_3)^2 + (c_4 + d_4)^2 g''(\xi_4)}{f} \\
&- \frac{[2a_1\xi_1 - 4c_2\xi_2 - 2(c_3 + d_3)\xi_3 - (c_4 + d_4)g'(\xi_4)]^2}{f^2}.
\end{aligned} \tag{2.6}$$

The above results supplement the existing theories of rational, soliton and dromion-type solutions obtained earlier by using Hirota perturbation technique [15], symmetry reductions [4, 10, 34], symmetry constraints [3, 11, 12, 49], multiple exp-function methods [13] and the Riemann-Hilbert technique [33].

In particular, considering the following set of parameters

$$\begin{aligned}
&a_1 = 1, \quad && b_2 = 2, \quad && d_2 = -1, \\
&b_3 = 3, && c_3 = -8, && d_3 = 5, \\
&b_4 = -5, && c_4 = 7, && d_4 = -6, \\
&\beta_1 = 1, && \beta_2 = 5, && \beta_3 = 6, \quad && \beta_4 = 15,
\end{aligned}$$

we obtain specific solutions u_i, $i = 1, 2, 3$ of the Eq. (2.5) — viz.

$$\begin{aligned}
u_1 &= \frac{28 f_1 - (28x + 26y - 52z + 24t)^2}{f_1^2}, \\
f_1 &= (x - t)^2 + (2x + 2y - z - t)^2 + (3x + 3y - 8z + 5t)^2 + 1, \\
u_2 &= \frac{(28 + 5\sin\xi_4) f_2 - (28x + 26y - 52z + 24t - 5\sin\xi_4)^2}{f_2^2}, \\
f_2 &= (x - t)^2 + (2x + 2y - z - t)^2 + (3x + 3y - 8z + 5t)^2 + 5\sin\xi_4 + 6, \\
u_3 &= \frac{(28 + 15\cosh\xi_4) f_3 - (28x + 28y - 52z + 24t + 15\sinh\xi_4)^2}{f_3^2}, \\
f_3 &= (x - t)^2 + (2x + 2y - z - t)^2 + (3x + 3y - 8z + 5t)^2 + 15\cosh\xi_4,
\end{aligned}$$

where $\xi_4 = x + 5y - 7z + 6t$. The graphs of these solutions are presented in Figs. 1, 2, 3.

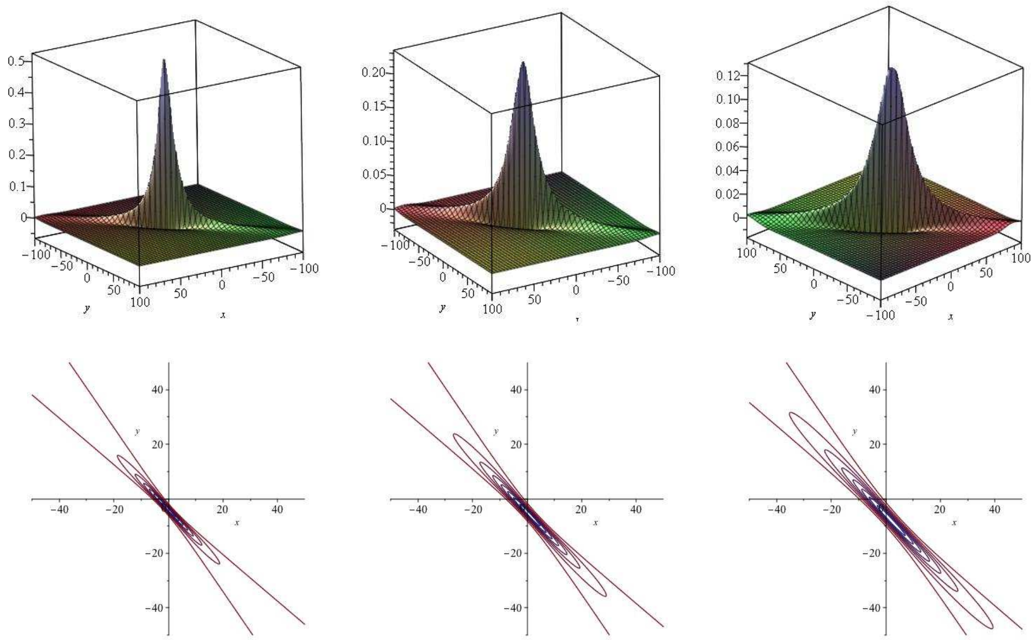

Figure 1: Profile of u_1, $t = 0, 1, 2$, $z = -2$. Top: 3d plots. Bottom: Contour plots.

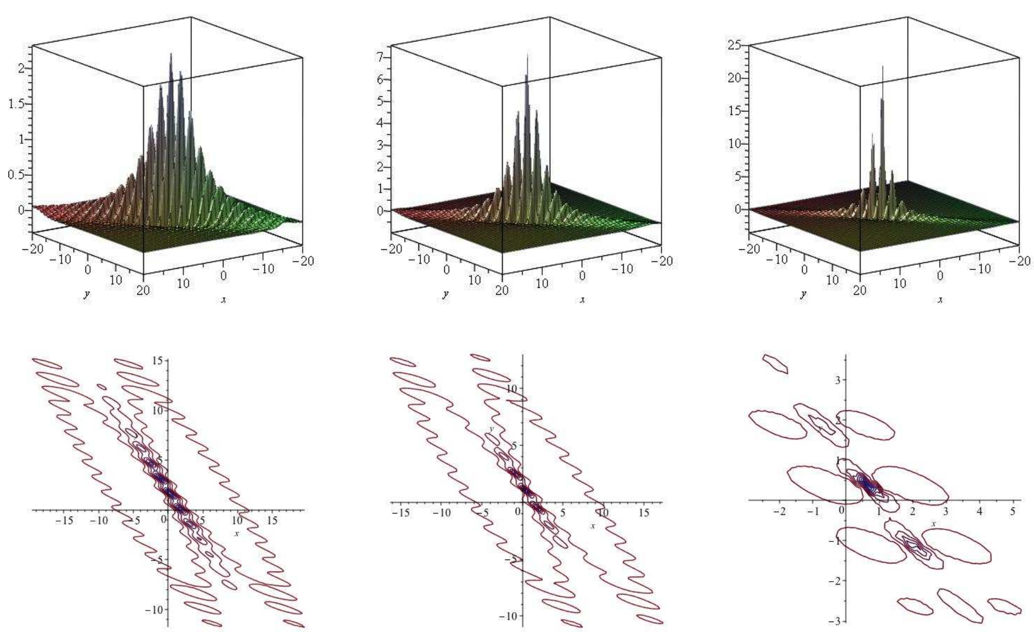

Figure 2: Profile of u_2, $t = 0, 0.5, 1$, $z = 1$. Top: 3d plots. Bottom: Contour plots.

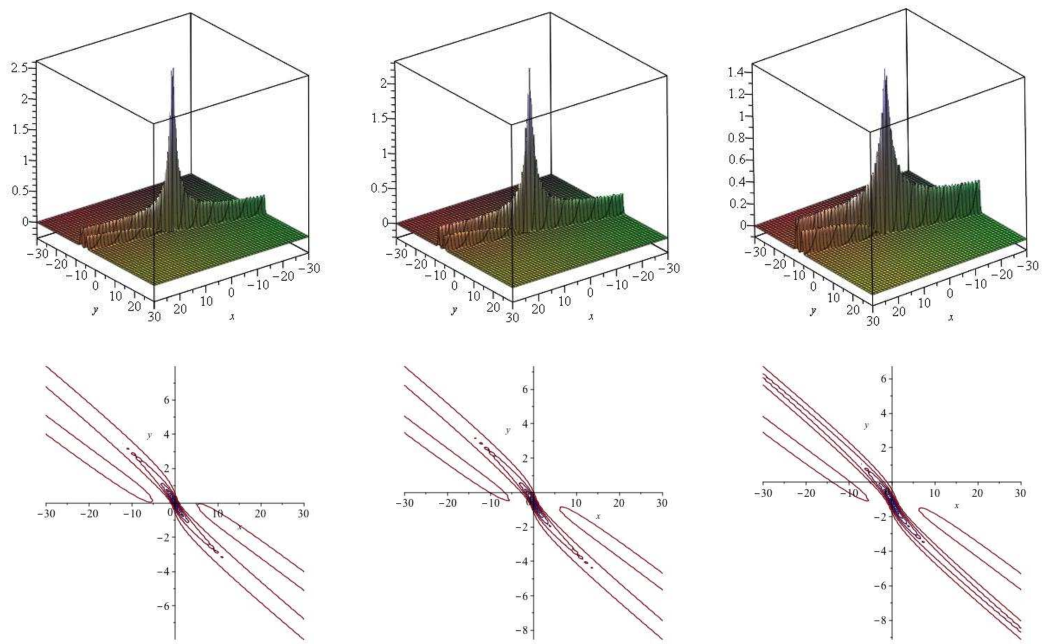

Figure 3: Profile of u_3, $t=0,0.5,1$, $z=0$. Top: 3d plots. Bottom: Contour plots.

3. Concluding Remarks

We considered specific linear partial differential equations in $(3+1)$-dimensions and showed that they have lump and interaction solutions such as lump-periodic, lump-kink and lump-soliton solutions, providing a new insight into soliton theory of integrable equations. The Maple symbolic computations were used to construct exact lump and interaction solutions of the considered equations in $3+1$ dimensions.

We observe that (2.4) and (2.6) with $g=0$ are lump solutions, rationally localised in all directions in the (x,y,z)-space. However, we were not able to find any analytical rational solutions of the considered linear PDEs, localised in all directions in the whole (x,y,z,t)-space. The lump and interaction solutions obtained above supplement the set of exact solutions which can be constructed by using various combinations in [23, 32, 36]. Lumps and interaction solutions of generalised bilinear and tri-linear equations involving generalised bilinear derivatives [17, 18] are also interesting, and the corresponding interaction solutions will not be the resonant solutions obtained by the linear superposition principles in [19,20]. Integrable equations determined by generalised bilinear derivatives [17,18] will have different interaction solutions, but lump solutions generated by quadratic functions must coincide with those in the Hirota derivative case — cf. Ref. [28]. Besides, there are also Rossby wave solutions of the generalised Boussinesq and Benjamin-Ono equations [14,37].

The diversity of lump and interaction solutions implies the existence of diverse Lie-Bäcklund symmetries, thus extending the symmetry theory of partial differential equations.

It is known that the Wronskian approach can be used to find solutions of integrable equations. The present study raises the problem of how to generalise the Wronskian solutions by introducing matrix entries of a new type. Moreover, it would also be of interest to develop a basic theory of lump and interaction solutions of difference-differential equations.

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