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Compound PDE-Based Additive Denoising Solution Combining an Improved Anisotropic Diffusion Model to a 2D Gaussian Filter Kernel

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Abstract. A second-order nonlinear anisotropic diffusion-based model for Gaussian additive noise removal is proposed. The method is based on a properly constructed edge-stopping function and provides an efficient detail-preserving denoising. It removes additive noise, overcomes blurring effect, reduces the image staircasing and does not generate multiplicative noise, thus preserving boundaries and all the essential image features very well. The corresponding PDE model is solved by a robust finite-difference based iterative scheme consistent with the diffusion model. The method converges very fast to the model solution, the existence and regularity of which is rigorously proved.

AMS subject classifications: 35Bxx, 94A08, 35K55, 35K60, 35Qxx, 65Nxx

Key words: Image restoration, nonlinear anisotropic diffusion, qualitative properties of solutions, boundary value problems for nonlinear parabolic PDE, Leray-Schauder principle.

1. Introduction

Image denoising is one of fundamental tasks in image processing. However, the classic $2D$ image filters often produce undesired blurring, which affects the edges and other essential image details [10], so that a feature-preserving restoration still represents a serious challenge.

The nonlinear partial differential equations (PDEs) have been increasingly used in the image denoising and restoration in the last three decades. They provide a good solution to the problem since Perona and Malik [18] introduced their celebrated anisotropic diffusion algorithm. Since then, various nonlinear second-order diffusion-based restoration models have been developed — cf. Refs. [5,23]. On the other hand the total variation (TV) denoising scheme proposed by Rudin *et al.* [19], initiated the development of numerous PDE variational filtering techniques [1,3,7,8,11,15,20,22].

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However, although such second-order PDE-based methods remove image blurring and preserve boundaries, they may cause another unintended problem — viz. staircase (or blocky) effect [6]. In contrast, nonlinear fourth-order diffusion-based models inspired by the influential You-Kaveh scheme [24], can successfully remove additive Gaussian noise and overcome the staircase effect — cf. [2,4]. Nevertheless, the over-filtering of the fourth-order diffusion models often affects the image and may produce undesired multiplicative speckle noise.

In this work, we develop a novel PDE-based technique, which successfully removes the additive noise, while avoiding or alleviating all the unintended effects mentioned. It is based on an improved second-order anisotropic diffusion model and a two-dimension Gaussian filter kernel. This model is discussed in Section 2 below. Section 3 deals with a fast-converging numerical approximation scheme based on the finite difference method in [9, 12] and consistent with the model under consideration. A rigorous mathematical analysis of this PDE-based model is provided in Section 4. In particular, we prove the existence and regularity of the classical solution for the corresponding nonlinear second-order diffusion-based scheme. In Section 5, we demonstrate the effectiveness of this restoration approach and compare it with other denoising models using the image quality measures [21]. Our conclusions are in Section 6.

2. A Nonlinear Second-Order Anisotropic Diffusion Model

We consider a novel second-order anisotropic diffusion-based model, which provides an effective detail-preserving image restoration. It is based on a boundary value problem for a nonlinear PDE — viz.

$$
\begin{aligned}
&\frac{\partial}{\partial t}u - \eta_u\left(\|\nabla G_\sigma * u\|\right)\nabla\cdot\left(\Psi^u(\|\nabla u\|)\nabla u\right) + \alpha(u-u_0) = 0,\\
&u(0,x_1,x_2) = u_0(x_1,x_2), \quad (x_1,x_2)\in\Omega\subset R^2,\\
&u(t,x_1,x_2) = 0, \quad t\in[0,T], \quad \forall(x_1,x_2)\in\partial\Omega,
\end{aligned}
\tag{2.1}
$$

where $\alpha\in(0,1]$, u_0 is the observed image, G_σ the Gaussian kernel,

$$
G_\sigma(x_1,x_2) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{x_1^2+x_2^2}{2\sigma^2}\right)
$$

determined by the standard deviation parameter $\sigma>0$, and $\|\cdot\|$ refers to the L^2 norm.

The function $\eta_u:(0,\infty)\to(0,\infty)$ in the PDE-based model (2.1) has the form

$$
\eta_u(s) = \frac{\left(\lambda s^k+\nu\right)^{1/(k+1)}}{\xi},
$$

where $\lambda,\nu\in(0,4]$, $\xi\geq 1.5$, $k\in(0,2]$. It is worth noting that the term $\eta_u\left(\|\nabla G_\sigma * u\|\right)$ controls the speed of this diffusion process and enhances the edges of the corresponding image.