East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.260318.220618

Vol. **9**, No. 1, pp. 45-66
February 2019

Compact Difference Scheme for Time-Fractional Fourth-Order Equation with First Dirichlet Boundary Condition

Mingrong Cui*

School of Mathematics, Shandong University, Jinan 250100, Shandong, China.

Received 26 March 2018; Accepted (in revised version) 22 June 2018.

Abstract. The convergence of a compact finite difference scheme for one- and two-dimensional time fractional fourth order equations with the first Dirichlet boundary conditions is studied. In one-dimensional case, a Hermite interpolating polynomial is used to transform the boundary conditions into the homogeneous ones. The Stephenson scheme is employed for the spatial derivatives discretisation. The approximate values of the normal derivative are obtained as a by-product of the method. For periodic problems, the stability of the method and its convergence with the accuracy $\mathscr{O}(\tau^{2-\alpha}) + \mathscr{O}(h^4)$ are established, with the similar error estimates for two-dimensional problems. The results of numerical experiments are consistent with the theoretical findings.

AMS subject classifications: 35R11, 65M06, 65M12, 65M15

Key words: Fractional partial differential equation, compact finite difference scheme, fourth-order equation, Stephenson scheme, stability and convergence.

1. Introduction

Let ${}_0^C D_t^\alpha v$, $0 < \alpha < 1$ be the Caputo fractional derivative of a function $v(x,t)$ [30] defined by

$$ {}_0^C D_t^\alpha v(x,t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial v}{\partial \tau}(x,\tau) d\tau. \tag{1.1} $$

We consider the one-dimensional time fractional fourth-order equation

$$ {}_0^C D_t^\alpha v(x,t) + \frac{\partial^4 v}{\partial x^4}(x,t) = g(x,t), \quad x \in (x_L, x_R), \quad t \in (0,T] \tag{1.2} $$

with the initial and boundary conditions

*Corresponding author. *Email addresses:* `mrcui@sdu.edu.cn` (Mingrong Cui)

$$\begin{aligned} &v(x,0) = v_0(x), \quad x \in [x_L, x_R] \\ &v(x_L,t) = \phi_L(t), \quad \frac{\partial v}{\partial x}(x_L,t) = \psi_L(t), \\ &v(x_R,t) = \phi_R(t), \quad \frac{\partial v}{\partial x}(x_R,t) = \psi_R(t). \end{aligned} \tag{1.3}$$

Considering the above problem in the situation where the time variable t does not appear, we obtain the biharmonic equation, which finds applications in incompressible fluid dynamics and in two-dimensional elasticity theory [2]. Numerical schemes for such problems are developed and well studied. Thus, for multi-space nonlinear parabolic partial differential equations and vibration problems, implicit difference schemes of order two in time and order four in space are, respectively, presented in [27] and [28]. It was already noted that for fourth-order diffusion equation with the second Dirichlet boundary conditions — i.e. if a second order derivative appears in the boundary conditions, the finding of numerical solutions is relatively easy. Thus writing the second order derivative as an auxiliary variable, one can split the original problem into a coupled system of two second-order equations with appropriate boundary conditions. However, the discretisation of the first Dirichlet boundary conditions requires special attention in order to match the global accuracy. As an uncoupled scheme, the Stephenson schemes of second and fourth order have been presented in [34], fourth order accurate schemes in [4] and [12], and a compact discretisation of the biharmonic problem with a fast FFT algorithm in [3].

Traditional partial differential equations contains the derivatives of integer order only. Recently, fractional differential equations attracted substantial attention because of wide applications — cf. [26,29,30]. Thus for anomalous subdiffusion equations, finite difference schemes are considered in Refs. [5,25,41,45]. Moreover, a difference scheme with spectral method [24] and fast finite difference methods [6, 37] are applied to space-fractional diffusion equations, to tempered fractional diffusion equations [17], to time fractional equations [22, 39] and to multi-term time-fractional diffusion equations [31]. Compact finite difference schemes for subdiffusion equations are proposed and studied in [1,8,14], where the error estimates $\mathcal{O}(\tau + h^4)$, $\mathcal{O}(\tau^{2-\alpha} + h^4)$, $\mathcal{O}(\tau^2 + h^4)$ are, respectively, obtained. For one-dimensional space and time fractional Bloch-Torrey equation, the stability and convergence of a high-order difference scheme have been studied in [44] by the discrete energy method. Various high-order difference schemes for Stokes' first problem are considered for heated generalized second grade fluid with fractional derivatives [21] and for distributed-order time-fractional equations [11]. Galerkin and spectral element methods for fractional equations have been investigated in [23,32,40].

The numerical solutions of fractional equations of fourth-order have been also considered — e.g. a compact algorithm for sub-diffusion equations with the first Dirichlet conditions [20]. A new variable was introduced and a high order difference scheme was developed with the convergence order $\mathcal{O}(\tau^{2-\alpha}+h^4)$ in L_2-norm. In addition, a local discontinuous Galerkin method for time-fractional fourth-order differential equations was studied in [16,38], an implicit compact finite difference scheme for the fourth-order fractional diffusion-wave system in [19], and the hyperbolic equation describing the random vibra-