East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.030118.070518

Vol. **9**, No. 1, pp. 67-86
February 2019

Multistep Collocation Methods for Volterra Integral Equations with Weakly Singular Kernels

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Received 3 January 2018; Accepted (in revised version) 7 May 2018.

Abstract. A multistep collocation method on uniform grids for weakly singular Volterra integral equations is considered. The initial integral equation is modified by a smoothing transformation and then solved by a multistep collocation method on a uniform grid. Convergence and linear stability are also studied. Numerical results demonstrate the efficiency of the method.

AMS subject classifications: 65L20

Key words: Multistep collocation method, Volterra integral equation, weakly singular kernel.

1. Introduction

Let $0 < \alpha < 1$ and $D := \{(t,\tau) : 0 \leqslant \tau \leqslant t \leqslant T\}$. Linear weakly singular Volterra Integral Equations (VIEs)

$$y(t) = g(t) + \int_0^t K(t,\tau)y(\tau)\mathrm{d}\tau, \quad t \in [0,T], \tag{1.1}$$

where $K(t,\tau) = (t-\tau)^{-\alpha}k(t,\tau)$, arise in many practical applications, including combustion, viscoelasticity, control theory, finance, biology, optics and so on — cf. [16, 19, 24, 25]. Such equations are well-studied and numerous results for the Eq. (1.1) can be immediately extended on nonlinear VIEs

$$y(t) = g(t) + \int_0^t (t-\tau)^{-\alpha}k(t,\tau,y(\tau))\mathrm{d}\tau, \quad t \in [0,T],$$

if $k : D \times \mathbb{R} \to \mathbb{R}$ and $g : [0,T] \to \mathbb{R}$ are sufficiently smooth functions — cf. [5, 7, 11].

Various numerical methods for weakly singular VIEs have been also developed [10, 18, 20, 27]. In particular, while dealing with weak singularity the collocation methods exploit two main ideas. Thus one can apply standard methods on suitable non-uniform meshes and

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this approach was used in one-step collocation methods [5, 8, 15]. Nevertheless, the use of small subintervals at the start of the method can lead to substantial round-off errors in subsequent calculations. To overcome these difficulties, various smoothing transformations have been proposed in [1, 2, 17, 22, 23]. Thus Diogo *et al.* [16] regularised weakly singular VIEs by a transformation and applied spline collocation methods based on uniform meshes. We note that working with smoothing transformations on a uniform partitions is equivalent to the use of graded meshes.

On the other hand, multistep collocation methods attract a lot of interest nowadays. Such methods employ collocation polynomials, relying on data and collocation points from previous time steps. The use of such methods allows to achieve prescribed accuracy at a much lower computational cost than in classical one-step collocation methods [12]. Moreover, multistep collocation methods have good stability. We recall that two-step almost collocation methods for VIEs with regular kernels have been constructed in [9, 11, 12], but generalisation of this approach to VIEs with weakly singular kernels requires substantial additional work. Here we engage the transformation from [23] to extend the approach of Conte and Paternoster [12] to weakly singular VIEs.

This paper is arranged as follows. Section 2 is devoted to the regularity of the solutions of the Eq. (1.1). In Section 3, multistep collocation methods for weakly singular VIEs are introduced and the order of convergence of such methods is established. In Section 4, we analyse the stability of multistep methods. Finally, Section 5 presents the results of numerical experiments, which illustrate the theoretical findings.

2. Regularity of the Solutions

Here and in what follows, C denotes a positive constant, independent of specified variables but taking different values in different occurrences. Let us introduce the weight function

$$\Lambda_\theta(t) := \begin{cases} 1, & \text{for} \quad \theta < 0, \\ (1 + |\log t|)^{-1}, & \text{for} \quad \theta = 0, \\ t^\theta, & \text{for} \quad \theta > 0, \end{cases}$$

where $t \in (0, T]$. We also consider two sets of smooth functions needed to describe the high order derivatives of the solutions of the Eq. (1.1).

Definition 2.1 (cf. Refs. [14, 23]). If $m \in \mathbb{N}$ and $\alpha < 1$, then $C^{m,\alpha}[0, T]$ is the set of all m-times continuously differentiable functions $y : [0, T] \to \mathbb{R}$ such that

$$\left| y^{(j)}(t) \right| \leqslant C\Lambda^{-1}_{\alpha+j-1}(t), \quad j = 1, 2, \cdots, m.$$

Definition 2.2 (cf. Refs. [14, 23]). If $m \in \mathbb{N}$ and $\alpha < 1$, then $W^{m,\alpha}(D)$ is the set of all m-times continuously differentiable functions $K : D \to \mathbb{R}$ such that

$$\left| \left(\frac{\partial}{\partial t} \right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau} \right)^j K(t, \tau) \right| \leqslant C\Lambda^{-1}_{\alpha+i}(t - \tau), \quad i + j \leqslant m, \quad i, j \in \mathbb{N}^+.$$