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The Breakdown of Darboux's Principle and Natural Boundaries for a Function Periodised from a Ramanujan Fourier Transform Pair

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Abstract. Darboux's Principle asserts that a power series or Fourier coefficient a_n for an analytic function $f(z)$ is approximated as $n \rightarrow \infty$ by a sum of terms, one for each singularity of $f(z)$ in the complex plane. This is crucial to understanding why Fourier series converge rapidly or slowly, and thus crucial to Fourier numerical methods. We partially refute Darboux's Principle by an explicit counterexample constructed by applying the Poisson Summation Theorem to a Fourier Transform pair found explicitly by Ramanujan. The Fourier coefficients show a geometric rate of decay proportional to $\exp(-\pi\chi\, n)$ multiplied by $\sin(\varphi)$ where the "phase" is $\varphi = \pi\chi^2 n^2 \mod (2\pi)$. We prove that the Fourier series converges everywhere within the largest strip centered on the real axis which is singularity-free, here $|\Im(z)| < \pi\chi$. We present strong evidence that the boundaries of the strip of convergence are natural boundaries. Because the function $f(z)$ is singular everywhere on the lines $\Im(z) = \pm\pi\chi$, there is no simple way to extrapolate the asymptotic form of the Fourier coefficients from knowledge of the singularities, as is possible through Darboux's Theorem when the singularities are isolated poles or branch points.

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1. Introduction

Fourier and Chebyshev polynomial spectral methods have become very popular in many fields of science and engineering [6–8, 10, 12, 13, 17–19, 24, 29, 30]. Shen, Tang and Wang [27] write "Along with finite differences and finite elements, spectral methods are one of the three main methodologies for solving partial differential equations on computers". Many practical issues arise in spectral methods, but the most fundamental is how rapidly or slowly the Fourier series converge. This makes convergence theory important to applied mathematics as well as being an ancient and beautiful area of pure mathematics.

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In passing note that the identity $T_n(\cos(t)) = \cos(nt)$ implies that a Chebyshev polynomial series is a Fourier cosine series with a change of variable, and therefore Chebyshev theory follows almost trivially from Fourier rate-of-convergence theory.

Much is known as summarised in the author's book [7] and Trefethen's monograph [30]. Much is still to be learned by studying interesting examples; broad theorems and numerical heuristics are solid only if built on a foundation of particular instances and illustrations. Here, we discuss examples that defied our expectations, the intuition from existing theory. In the process, we find a modern use for some neglected work of Ramanujan [1,21].

Fourier Transform theory, Darboux's Principle, and the Poisson Summation Theorem in its general imbricate series form are key technologies. Fear not if some of these are unfamiliar; we shall provide the necessary background as we shall go. Darboux's Principle is that each singularity of a function $f(z)$, and a similar factor for entire (integral) functions, makes its own unique and readily-calculable contribution to the asymptotic Fourier coefficients of the series for $f(z)$. The Poisson Summation Theorem as used here shows that every periodic function $f(z)$ can be represented as an infinite series ("imbricate series") of uniformly spaced copies of a "pattern function" or "imbrex" which is the Fourier Transform of the function $g(n)$ that gives the Fourier coefficients of $f(z)$.

Ramanujan published some novel Fourier integrals in 1915 and 1919 [25,26]. Watson wrote a second sequel in 1936 [32]. Titchmarsh reviewed their contributions in his 1937 book on Fourier integrals [28].

Ramanujan discovered four transform pairs, but we shall present only the following which is typical:

Lemma 1.1 (Ramanujan's Fourier Transform Pair)**.** *The function $G(z)$ defined below is the Fourier Transform of $g(n)$, where*

$$g(n) = \sin(\pi n^2)\,\mathrm{sech}\,(\pi n), \quad G(z) = \left\{\cos\left(\frac{z^2}{4\pi}\right) - \frac{1}{\sqrt{2}}\right\}\mathrm{sech}\,(z/2)$$

and the Fourier Transform is normalised so that

$$\int_{-\infty}^{\infty} \sin(\pi z^2)\,\mathrm{sech}\,(\pi z)\exp(ikz)\,dz = \left\{\cos\left(\frac{k^2}{4\pi}\right) - \frac{1}{\sqrt{2}}\right\}\mathrm{sech}\,(k/2).$$

Ramanujan and Watson went no further in the direction pursued here. However, substituting this pair into the Poisson Theorem [5,11,35] allows us to create an analytic periodic function of period P with the property that the Fourier coefficients $g(n)$ are given exactly by one of Ramanujan's functions and the pattern function that generates the second series below, is the Fourier Transform of the Fourier coefficient function $g(n)$,

$$\begin{aligned}\mathfrak{F}(z;\chi) &\equiv \chi \sum_{n=-\infty}^{\infty} \sin(\pi\chi^2 n^2)\,\mathrm{sech}\,(\pi\chi n)\exp\left(i\frac{2\pi}{P}nz\right) \\ &= \sum_{m=-\infty}^{\infty}\left\{\cos\left(\frac{2\pi}{P\chi}\frac{[z-mP]^2}{4\pi}\right) - \frac{1}{\sqrt{2}}\right\}\mathrm{sech}\,(\left(\frac{2\pi}{P\chi}\frac{(z-mP)}{2}\right).\end{aligned}$$