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Global Existence of Axisymmetric Pathwise Solutions for Stochastic Three-Dimensional Axisymmetric Navier-Stokes Equations

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Abstract. A stochastic three-dimensional Navier-Stokes system with the axisymmetric initial data and white noise is studied. It is shown that if the swirl component of the initial velocity field and the white noise are sufficiently small, then the axisymmetric pathwise solution is global in probability. Moreover, in the absence of the swirl, the pathwise axisymmetric solution is global almost surely.

AMS subject classifications: 35Q35, 60H15, 76B03

Key words: Stochastic Navier-Stokes equations, axisymmetric, global existence.

1. Introduction

We consider the initial value problem for the three-dimensional Navier-Stokes equation

$$
\begin{aligned}
&\partial_t u + (u\cdot\nabla)u - \Delta u + \nabla p = f, \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3,\\
&\nabla\cdot u = 0,\\
&u|_{t=0} = u_0,
\end{aligned}
\tag{1.1}
$$

where $u(t,x)$ is the fluid velocity, $p(t,x)$ pressure and $f(t,x)$ a bulk force. If $f\in H^{-1}(\mathbb{R}^3)$, $u_0\in L^2(\mathbb{R}^3)$ and $\operatorname{div} u_0 = 0$ in the sense of distributions, Leray [24] constructed a global finite energy weak solution of the problem (1.1) — cf. also [12]. Nevertheless, the uniqueness and the regularity of such solutions are still open questions [7, 22]. In particular, the axisymmetric solutions of the three-dimensional Navier-Stokes systems are of special interest in fluid mechanics. Let us remind that the solution of (1.1) is called axisymmetric, if in the cylindrical coordinate system it can be represented in the form

$$
u(t,x) = u^r(t,r,x_3)e_r + u^\theta(t,r,x_3)e_\theta + u^3(t,r,x_3)e_3, \tag{1.2}
$$

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where

$$e_r=\Big(\frac{x_1}{r},\frac{x_2}{r},0\Big),\quad e_\theta=\Big(-\frac{x_2}{r},\frac{x_1}{r},0\Big),\quad e_3=(0,0,1),\quad r=\sqrt{x_1^2+x_2^2}.$$

If the swirl is absent — i.e. if $u^\theta=0$, the global well-posedness of axisymmetric solutions was independently proved by Ukhovskii and Yudovich [28] and Ladyzhenskaya [19]. For a refined proof, the reader can consult [23]. However, if $u^\theta\neq 0$, the global well-posedness is still an open problem, although a substantial progress has been made in the study of the regularity of the solutions — cf. Refs. [2–4, 6, 15–17, 20, 27]. Moreover, P. Zhang and T. Zhang [30] investigated the global well-posedness if the swirl component u_0^θ of the initial velocity field is small, Chen *et al.* [5] proved that the axisymmetric solution is global if $u_0^\theta\in L^3(\mathbb{R}^3)$ is sufficiently small. More related results can be found in [21, 29].

Stochastic three-dimensional Navier-Stokes equations also attracted substantial attention recently. On the one hand, the stochastic models have the ability to represent both thermodynamic fluctuations and small perturbations — e.g. numerical, empirical, and physical uncertainties. On the other hand, this approach allows to describe the turbulence in the statistical sense — cf. [10, Chapter 4] and references therein. Stochastic three-dimensional Navier-Stokes equations have the form

$$\begin{aligned}
&du+(u\cdot\nabla u-\Delta u+\nabla p)\,dt=\sum_{i=1}^{\infty}\sigma_i dW_i(t),\quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3,\\
&\nabla\cdot u=0,\\
&u|_{t=0}=u_0,
\end{aligned}\tag{1.3}$$

where W_i, $i=1,2,\cdots$ is a collection of one-dimensional independent Brownian motions defined in the filtered space $(\Omega,\mathscr{F},\mathscr{F}_t,\mathbb{P})$ and $\mathscr{F}_t$ are sub σ-fields of $\mathscr{F}$ such that $\mathscr{F}_s\subset\mathscr{F}_t\subset\mathscr{F}$ for $0\le s<t<\infty$. Note that the pathwise and martingale solutions of (1.3) are considered in [14, 26]. However, to the best of our knowledge, the axisymmetric solutions of stochastic three-dimensional Navier-Stokes equations have not been yet studied. Here we consider the pathwise solution of (1.3) such that

$$\begin{aligned}
&u_0(t,x)=u_0^r(r,x_3)e_r+u_0^\theta(r,x_3)e_\theta+u_0^3(r,x_3)e_3,\\
&\sigma_i(t,x)=\sigma_i^r(t,r,x_3)e_r+\sigma_i^\theta(t,r,x_3)e_\theta+\sigma_i^3(t,r,x_3)e_3
\end{aligned}$$

for $1\le i\le\infty$. More exactly, we establish the global existence of axisymmetric pathwise solutions in probability if the initial data and the random noise are sufficiently small. Let us consider the spaces

$$\begin{aligned}
&\mathscr{H}^m:=\{f\,|\,f\in H^m, f \text{ is axisymmetric}\},\\
&\mathscr{H}_0^m:=\{f\,|\,f\in\mathscr{H}^m,\nabla\cdot f=0\},
\end{aligned}$$

equipped with the norm

$$\|f\|_{H^m}^2=\sum_{k=0}^{m}\|\nabla^k f\|_{L^2}^2$$