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High-Order Energy-Preserving Methods for Stochastic Poisson Systems

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Abstract. A family of explicit parametric stochastic Runge-Kutta methods for stochastic Poisson systems is developed. The methods are based on perturbed collocation methods with truncated random variables and are energy-preserving. Under certain conditions, the truncation does not change the convergence order. More exactly, the methods retain the mean-square convergence order of the original stochastic Runge-Kutta method. Numerical examples show the efficiency of the methods constructed.

AMS subject classifications: 60H10, 65C20, 37N30

Key words: Stochastic Poisson systems, stochastic Runge-Kutta methods, energy-preserving, mean-square convergence.

1. Introduction

Stochastic differential equations (SDEs) have been used to model various physical, financial and biological phenomena [17]. Except for special cases, explicit analytic solutions of such equations are not known, so that various numerical methods for SDEs have been developed in recent decades — cf. Refs. [2, 3, 9, 12, 13, 16, 19, 28].

Since many systems have important physical or geometric properties, it is natural to use approximation methods that preserve the peculiarities of the systems. In this paper we are concerned with energy conservation – a significant property of mechanical systems. Energy-preserving numerical methods have proven to be more stable than general ones, especially in long-term simulations. Numerous energy-preserving numerical methods for deterministic systems can be found in literature — cf. [1, 4, 5, 7, 10, 18, 23–26]. Nevertheless, energy-preserving methods for stochastic systems are less developed, although for stochastic ordinary differential equations one can note difference methods [21], discrete

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gradient methods [11, 15], projection methods [15, 27], averaged vector field methods [6] and a few others.

Here, we consider the stochastic Poisson systems in Stratonovich sense

$$\begin{aligned} &dx = B(x)\nabla H(x)(dt + \sigma \circ dW_t), \quad 0 \le t \le T, \\ &x(0) = x_0, \end{aligned} \tag{1.1}$$

where x is an n-dimensional column vector, $\sigma \in \mathbb{R}$ a constant, and W_t a 1-dimensional standard Wiener process defined on a complete filtered probability space $(\Omega, \mathscr{F}, \mathscr{P}, \{\mathscr{F}_t\}_{t\ge 0})$— cf. [6]. Besides, $H(x)$ is a real-valued energy function, $B(x) \in \mathbb{R}^{n\times n}$ a skew-symmetric matrix-valued function, x_0 a $\mathscr{F}_0$-measurable random variable with $E\|x_0\|^2 < \infty$ and $\|\cdot\|$ the Euclidean norm. If I is an identity matrix and

$$B(x) = J^{-1} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

then (1.1) reduces to a stochastic Hamiltonian system.

The results of this work can be easily extended to the following stochastic Poisson systems with multiple Wiener processes

$$\begin{aligned} &dx = B(x)\nabla H(x)\left(dt + \sum_{i=1}^{r} \sigma_i \circ dW_t^i\right), \quad 0 \le t \le T, \\ &x(0) = x_0, \end{aligned} \tag{1.2}$$

since the notations

$$\sigma = \left(\sum_{i=1}^{r} \sigma_i^2\right)^{1/2}, \quad W_t = \frac{1}{\sigma}\sum_{i=1}^{r} \sigma_i W_t^i$$

transform (1.2) into (1.1) [8].

It is easy to verify that the energy function $H(x)$ is a conserved quantity for the system (1.1). To find the solution of the system (1.1), Cohen and Dujardin [6] proposed energy-preserving stochastic averaged vector field methods. However, these methods have only mean-square convergence order 1 as many other numerical approaches to stochastic problems. In contrast, the energy-preserving numerical methods developed in this work, can achieve arbitrary high convergence order. It is also worth mentioning that most of the existing stochastic energy-preserving numerical approaches, including discrete gradient and averaged vector field methods are fully implicit — i.e. at each step they require to find solutions of high dimensional nonlinear systems that inevitably increases computational cost. On the other hand, the energy-preserving numerical methods considered here are based on explicit parametric stochastic Runge-Kutta (SRK) methods. Therefore, in order to determine the corresponding parameter, at each step only one nonlinear equation has to be solved and the computational cost is much lower.

The rest of the paper is organised as follows. Section 2 reviews perturbed collocation methods for ordinary differential equations (ODEs). In Section 3, we develop a family of