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Generalised Backward Differentiation Formulae for Fractional Differential Equation

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Abstract. A new approach to construction of generalised backward differentiation formulae for fractional differential equations is proposed. The consistence and convergence of the method are considered and numerical simulations are presented.

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Key words: Fractional differential equation, boundary value method, fractional linear multistep method, consistency, convergence.

1. Introduction

Fractional differential equations (FDE) become an important tool in modelling various phenomena [8, 15, 27, 33]. Here we consider the scalar FDE

$$\begin{aligned} &{}_{t_0}^{C}D^{\beta}u(t) = g\left(t, u(t)\right), \quad t_0 \leqslant t \leqslant T, \\ &u(t_0) = u_0, \end{aligned} \tag{1.1}$$

where $0 < \beta < 1$, $g : [t_0, T] \times \mathbb{C} \to \mathbb{C}$ and ${}_{t_0}^{C}D^{\beta}u(t)$ is the fractional derivative in the Caputo sense — cf. [25], i.e.

$${}_{t_0}^{C}D^{\beta}u(t) = \frac{1}{\Gamma(1-\beta)} \int_{t_0}^{t} \frac{u'(s)}{(t-s)^{\beta}} \mathrm{d}s.$$

If the function g is continuous and satisfies the Lipschitz condition with respect to the second variable, the problem (1.1) has a unique solution [7].

The solutions of most FDEs are not known, so that a variety of approximation methods have been developed, including finite difference scheme [14, 23, 29], Galerkin finite element method [16, 32], separable preconditioner [18], $L1$-approximation scheme [17, 30], Galerkin spectral method [26] and parareal algorithms [28]. Lubich [19, 21, 22] started the development of fractional linear multistep methods (FLMMs). In contrast, Galeone and

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Garrappa [9] proposed another approach to FLMMs via the expansion of local truncation errors. Nevertheless, the above methods still have a number of deficiencies. The resulting schemes are prone to heavy order limitation for A-stable methods. Lubich [20] extended the famous concept of classical second Dahlquist barrier of linear multistep methods (LMMs) for ordinary differential equations (ODE) [5] to fractional case.

For ODE, this barrier can be overcome if the discrete problem generated by LMMs is completed by imposing boundary value methods (BVMs) because BVMs have a good stability and high accuracy — cf. [3,4,12]. This motivated numerous researches for corresponding fractional BVMs and FBVMs. In particular, the authors of this paper, used generalised Adams methods (GAMs) to introduce a family of fractional convolution quadratures [1,2]. It was shown that such methods can overcome classical order barrier. More recently, an A_0-stable high order fractional backward differentiation formulae has been considered [31]. Here, we employ generalised backward differentiation formulae (GBDF) to construct a new type of FBVMs for the FDE (1.1).

This paper is organised as follows. In Section 2, we present general FBVMs for FDE (1.1) and introduce the notion of the consistence of the methods proposed. In Section 3, a fractional GBDF (FGBDF) is constructed and its convergence is analysed. Numerical examples presented in Section 4 illustrate theoretical results.

2. Consistence of FBVMs

Let us start with the k-order n-step FLMMs for the Eq. (1.1) introduced in [9,10] — viz. we consider the system of equations

$$\sum_{j=0}^{n}\alpha_{k,j}u_{n-j}=h^{\beta}\sum_{j=0}^{n}\gamma_{k,j}g_{n-j}, \tag{2.1}$$

where $\alpha_{k,j}$ and $\gamma_{k,j}$ are constants subject to the order conditions, u_n is an approximation to $u(t_n)$ at the point $t_n=t_0+nh$ and $g_n=g(t_n,u_n)$.

Moreover, we define an operator $\mathscr{L}_h$ by

$$\mathscr{L}_h[u(t),t,\beta]:=C_0(n,\beta)u(t_0)+\sum_{p=1}^{k}h^pC_p(n,\beta)u^{(p)}(t_0)+h^{k+1}R_{k+1},$$

where R_{k+1} is the remainder in the Taylor expansion of the function $u(t)$ and

$$\begin{aligned}C_0(n,\beta)&:=\sum_{j=0}^{n}\alpha_{k,j},\\ C_p(n,\beta)&:=\frac{1}{p!}\sum_{j=0}^{n}(n-j)^p\alpha_{k,j}-\frac{1}{\Gamma(p+1-\beta)}\sum_{j=0}^{n}(n-j)^{p-\beta}\gamma_{k,j},\quad p=1,2,\cdots,k.\end{aligned}$$

In order to determine the convergence order, we set

$$C_p(n,\beta)=0,\quad p=0,1,\cdots,k. \tag{2.2}$$