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Convolution Quadrature Methods for Time-Space Fractional Nonlinear Diffusion-Wave Equations

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Abstract. Two second-order convolution quadrature methods for fractional nonlinear diffusion-wave equations with Caputo derivative in time and Riesz derivative in space are constructed. To improve the numerical stability, the fractional diffusion-wave equations are firstly transformed into equivalent partial integro-differential equations. Then, a second-order convolution quadrature is applied to approximate the Riemann-Liouville integral. This deduced convolution quadrature method can handle solutions with low regularity in time. In addition, another second-order convolution quadrature method based on a new second-order approximation for discretising the Riemann-Liouville integral at time $t_{k-1/2}$ is constructed. This method reduces computational complexity if Crank-Nicolson technique is used. The stability and convergence of the methods are rigorously proved. Numerical experiments support the theoretical results.

AMS subject classifications: 65M06, 65M12, 35R1

Key words: Fractional diffusion-wave equation, nonlinear source, convolution quadrature, generating function, stability and convergence.

1. Introduction

Fractional partial differential equations naturally arise in anomalous diffusion with random walk processes due to non-local properties of fractional integrals and fractional derivatives. The fractional anomalous diffusion models are obtained by replacing the integer-order calculus operators in the classical diffusion equations by fractional operators for

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the memory effects [26, 34, 37, 42]. In particular, time and space fractional diffusion-wave equations can interpolate the diffusion and the wave phenomena and describe processes with spatial non-local dependence. Therefore, such models are widely used for description of viscoelastic damping materials, diffusion images of human brain tissues, etc. [11,16,22,29,35].

Since it is very difficult or often impossible to obtain analytical solutions of fractional diffusion-wave equations [1,3,25,27,28], numerical methods are required. There is a vast literature on approximation methods for time or time-space fractional linear diffusion-wave equations [2, 5, 6, 12, 13, 19–21, 30, 33, 36, 38, 41, 43–45]. Using a classical $(3-\alpha)$-order approximation for the Caputo derivative, Sun and Wu [38] constructed a finite difference scheme and studied its stability and convergence. Li *et al.* [19] applied a finite difference method in time and finite element method in space to time-space fractional diffusion-wave equations and investigated semidiscrete and fully discrete numerical approximations. Liu *et al.* [21] considered numerical methods for multi-term time-fractional wave-diffusion equations. Using equivalent partial integro-differential equations, Huang *et al.* [13] constructed two finite difference schemes for a class of time fractional diffusion-wave equations and proved their first- and second-order convergence in temporal and spatial directions, respectively. Mustapha and Schötzau [30] established the well-posedness of an *hp*-version of time-stepping discontinuous Galerkin method for fractional diffusion-wave evolution problems, derived error estimates in a nonstandard norm and showed exponential convergence in the number of temporal degrees of freedom for solutions with singular behavior near $t=0$.

On the other hand, Wang and Vong [41] used a weighted and shifted Grünwald difference operator and compact difference technique to construct a higher order scheme for a time fractional diffusion-wave equation. Bhrawya *et al.* [2] presented a spectral numerical method for fractional diffusion-wave and fractional wave equations with damping. The method is based on the Jacobi τ-spectral procedure and Jacobi operational matrix for fractional Riemann-Liouville integrals. Zeng [44] proposed second-order in time and space stable and conditionally stable finite difference schemes for time fractional super-diffusion equation based on the fractional trapezoidal rule and the generalised Newton-Gregory formula. Ye *et al.* [43] derived a compact difference scheme for a distributed-order time-fractional diffusion-wave equation, and proved its unique solvability, stability and convergence. Chen and Li [5] used equivalent integro-differential equations and product trapezoidal rule to construct a compact finite difference scheme for fractional diffusion-wave equations. Chen *et al.* [6] considered a second-order backward differentiation formula alternating direction implicit difference for two-dimensional time fractional diffusion-wave equations.

The problems arising in numerical solution of non-linear fractional diffusion-wave equations are more complex. Nevertheless, Dehghan and Abbaszadeh [7] constructed a finite difference-spectral element method and studied its stability and convergence. This method performs better than other existing methods. Huang and Yang [14] combined the spectral Galerkin method in space and the fractional trapezoid method in time having the spectral accuracy in space.