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Efficient Spectral Stochastic Finite Element Methods for Helmholtz Equations with Random Inputs

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Abstract. The implementation of spectral stochastic Galerkin finite element approximation methods for Helmholtz equations with random inputs is considered. The corresponding linear systems have matrices represented as Kronecker products. The sparsity of such matrices is analysed and a mean-based preconditioner is constructed. Numerical examples show the efficiency of the mean-based preconditioners for stochastic Helmholtz problems, which are not too close to a resonant frequency.

AMS subject classifications: 65C30, 65F08, 65N30, 35J05

Key words: Helmholtz equations, PDEs with random data, generalised polynomial chaos, stochastic finite elements, iterative solvers.

1. Introduction

During last decades there has been a rapid development in uncertainty quantification for solving partial differential equations (PDEs) with random inputs. These random inputs usually arise from lack of knowledge about the system or/and the measurements of realistic model parameters such as the permeability coefficients in diffusion problems [29, 55], the viscosity parameters of incompressible flows [12, 42, 47, 49], and shape parameters in acoustic scattering [58]. In particular, stochastic Helmholtz equations attracted a lot of interest recently and this paper aims at the development of efficient strategies for their solution.

The Helmholtz equation plays a fundamental role in ocean acoustics, optic and electromagnetic problems [28, 33, 37, 51]. In acoustics wave problems, the uncertainties come from the refractive indices (or wave number parameters), source functions, and the shapes of scattering surfaces. Elman *et al.* [14] considered Helmholtz equations with random forcing functions and boundary conditions and developed efficient multigrid solvers for the

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corresponding stochastic finite element approximations. Xiu and Shen [58] developed generalised polynomial chaos (gPC) approximations [55, 56] (for polynomial chaos see [30]) based on stochastic collocation methods [2, 54] to solve the problems with uncertain scattering surface shapes. Tang and Zhou [60] investigated a stochastic collocation method for scalar hyperbolic equations with a random wave speed and showed that the rate of convergence depends on the regularity of solutions. Later on, multifidelity approaches to stochastic optimisation problems, including stochastic wave numbers and impedance parameters have been studied [39, 40]. Papers [22, 23] deal with a Monte Carlo interior penalty discontinuous method. Fang *et al.* [21] developed a stochastic Galerkin method for Maxwell's equations with random input.

Here we study spectral stochastic Galerkin finite element methods [3,30,52] for stochastic Helmholtz equations with uncertainties in refractive indices. The stochastic parameter space is discretised by gPC methods of [55, 56] and the physical space by finite element methods of [6, 15]. This leads to linear systems in Kronecker formulation [10, 41, 43]. We note that for stochastic Galerkin linear systems, various efficient iterative solvers such as mean-based preconditioning methods [41, 43], hierarchical preconditioners [47, 48], preconditioned low-rank projection methods [36] are vigorously studied. Nevertheless, to the best of our knowledge, in the case of stochastic Helmholtz problems these methods have not been analysed. Here we investigate the sparsity of the stochastic Galerkin linear systems associated with stochastic Helmholtz problems and the corresponding mean-based preconditioning scheme.

The outline of this work is as follows. In Section 2, we describe the problem, introduce spectral stochastic Galerkin finite element approximations, discuss the sparsity of the underling linear systems and present the linear systems associated with uniform random inputs. In Section 3, iterative methods and mean-based preconditioning are discussed. Section 4 contains numerical results and our conclusions are in Section 5.

2. A Stochastic Helmholtz Equation and its Discretisation

Let $D \subset \mathbb{R}^d$, $d = 2,3$ denote a physical domain and $\boldsymbol{x} \in \mathbb{R}^d$ the physical variable. We assume that D is bounded, connected and has a polygonal boundary ∂D. Moreover, let $\xi = [\xi_1, \cdots, \xi_N]^T$ be the vector of real-valued random variables. The image of ξ is denoted by Γ and the probability density function of ξ by $\pi(\xi)$. Here, we consider the following stochastic Helmholtz problem: Find an unknown function $u(\boldsymbol{x}, \xi)$ such that

$$-\nabla^2 u(\boldsymbol{x}, \xi) - \kappa^2(\boldsymbol{x}, \xi) u(\boldsymbol{x}, \xi) = f(\boldsymbol{x}) \qquad \forall(\boldsymbol{x}, \xi) \in D \times \Gamma, \tag{2.1}$$

$$u(\boldsymbol{x}, \xi) = 0 \qquad \forall(\boldsymbol{x}, \xi) \in \partial D_D \times \Gamma, \tag{2.2}$$

$$\frac{\partial u}{\partial n} - \mathbf{i}\kappa(\boldsymbol{x}, \xi) u = 0 \qquad \forall(\boldsymbol{x}, \xi) \in \partial D_R \times \Gamma, \tag{2.3}$$

where $\kappa \in \mathbb{R}$ is the refractive index, $\partial u / \partial n$ the outward normal derivative of u on the boundary and $\mathbf{i} = \sqrt{-1}$. Moreover, we assume that the Dirichlet ∂D_D and the radiation