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SPREAD OPTION PRICING USING ADI METHODS

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Abstract. Spread option contracts are becoming increasingly important, as they frequently arise in the energy derivative markets, e.g. exchange electricity for oil. In this paper, we study the pricing of European and American spread options. We consider the two-dimensional Black-Scholes Partial Differential Equation (PDE), use finite difference discretization in space and consider Crank-Nicolson (CN) and Modified Craig-Sneyd (MCS) Alternating Direction Implicit (ADI) methods for timestepping. In order to handle the early exercise feature arising in American options, we employ the discrete penalty iteration method, introduced and studied in Forsyth and Vetzal (2002), for one-dimensional PDEs discretized in time by the CN method. The main novelty of our work is the incorporation of the ADI method into the discrete penalty iteration method, in a highly efficient way, so that it can be used for two or higher-dimensional problems. The results from spread option pricing are compared with those obtained from the closed-form approximation formulae of Kirk (1995), Venkatramanan and Alexander (2011), Monte Carlo simulations, and the Brennan-Schwartz ADI Douglas-Rachford method, as implemented in MATLAB. In all spread option test cases we considered, including American ones, our ADI-MCS method, implemented on appropriate non-uniform grids, gives more accurate prices and Greeks than the MATLAB ADI method.

Key words. Modified Craig-Sneyd, alternating direction implicit method, two-dimensional Black-Scholes, American option, spread option, exchange option, analytical approximation, numerical PDE solution, and penalty iteration.

1. Introduction

Spread options are popular financial contracts for which, except the simplest case called *exchange contracts* (i.e. strike = 0), there exist no analytical solutions. We are interested in pricing spread options, using a numerical Partial Differential Equation (PDE) approach.

A spread option is a two-asset derivative, whose payoff depends on the difference between the prices $s_1(t)$ and $s_2(t)$ of the two assets. Essentially, the European call (put) spread option gives the holder the right, but not the obligation, to buy (sell) the spread $s_1(T) - s_2(T)$ at the exercise price K, at maturity time T.

In the energy markets, an example of a spread option is an option on the *spark spread*, which is the difference between the price received by a generator for electricity produced and the cost of the natural gas needed to produce that electricity. Another example of a spread option is an option on the *crack spread*, which is the difference between the price of refined petroleum products and crude oil.

Several spread option pricing approaches can be used, such as Monte Carlo (MC) simulations or tree (lattice) methods, however, for problems in low dimensions, the PDE approach is a popular choice, due to its efficiency and global character. In addition, the essential parameters for risk-management and hedging of financial derivatives, such as delta and gamma, are generally much easier to compute via the PDE approach than via other methods.

We consider the basic two-dimensional Black-Scholes (BS) PDE and compute a numerical approximation to the solution using second-order finite differences (FD)

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discretization in space and Crank-Nicolson (CN) or Alternating Direction Implicit (ADI) methods, more specifically, the Modified Craig-Sneyd (MCS) method, for timestepping. Note that MCS is stable without stepsize restrictions, and exhibits second-order convergence in both space and time when applied to PDEs with mixed derivative terms ([12], [13], [24]). To price American spread options a non-linear penalty term is added to the PDE, and an iterative method is employed to solve the resulting problem. This technique is introduced in [8] for one-dimensional problems and CN timestepping. Our focus in this paper is to incorporate the ADI-MCS method efficiently into the penalty iteration. We present results from our ADI-MCS method on various types of spread options and compare them to results from other PDE approximation methods, analytical approximation formulae and MC simulation methods.

Regarding analytical formulae, for the case of European exchange options, there exists the *Margrabe* formula [17], which gives the exact price of the exchange option, and is an accurate reference value for comparison. For general European spread options, there is no formula that gives the exact price. However, there exist analytical formulae that approximate the price of European spread options. These approximations include Kirk's formula [15] and the formula developed by Venkatramanan and Alexander [22]. Kirk's formula provides a good approximation of spread option prices when the strike K is small compared to the current value of s_2. Venkatramanan and Alexander [22] express the price of a European spread option as the sum of the prices of two compound exchange options (CEOs), one to exchange vanilla call options and the other to exchange vanilla put options. Using a conditional relationship between the strike of vanilla options and the implied correlation, the authors reduce the problem to one-dimensional put and call vanilla option computations.

For both European and American spread option test cases, we also consider two different numerical approximations in MATLAB. One is MATLAB's implementation of the Douglas-Rachford (DR) ADI timestepping (func. `spreadsensbyfd`), and another is the MC simulations (func. `spreadsensbyls`). In the MATLAB function `spreadsensbyfd`, for American spread options, the ADI-DR timestepping is combined with the Brennan-Schwartz algorithm [1], in order to solve the Linear Complementarity Problem (LCP) arising when pricing American type options. In [10], the algorithm is re-formulated using a form of LU decomposition, and this form is used in the MATLAB code.

Moreover, we display results of Greeks computed using our method, as well as MATLAB's functions `spreadsensbyfd` and `spreadsensbyls`. Finally, we present results for the free boundary.

In Section 2, we present the PDE along with initial and boundary conditions. The discretization of the problem along the space dimension and the ADI timestepping technique are described in Section 3. In Section 4, the handling of American options using the discrete penalty iteration method for both CN and ADI timestepping is demonstrated. Special attention is paid in efficiently incorporating the ADI-MCS method into the discrete penalty iteration. Finally, in Section 5, we present numerical results that demonstrate the performance of our method. We first compare our results for a European exchange option with Margrabe's exact solution. We then make an experimental investigation on the accuracy and convergence of the price computed by our method for both uniform and non uniform