INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 16, Number 5, Pages 745–766

ANISOTROPIC MESH ADAPTATION METHOD BASED ON ANISOTROPIC BUBBLE-TYPE LOCAL MESH GENERATION

WEI GUO, YUFENG NIE AND WEIWEI ZHANG*

Abstract. A new anisotropic adaptive mesh refinement method based on anisotropic bubble-type local mesh generation (ABLMG) for elliptic partial differential equations is proposed. The anisotropic meshes are generated as quasi-uniform meshes in metric spaces with the metric determined on each vertex by anisotropic a posteriori error estimator. Under the new metric, the error is equidistributed in the directions of maximum and minimum stretching on an element, and the mesh size is reduced/coarsend in regions with large/small errors. With the full use of the adjacent lists provided by the node placement method, the local mesh for each vertex is generated through ABLMG method. Compared with other methods, the mesh refining and coarsening can be obtained in the same framework and the mesh suits the metric well at each refinement level. Numerical results in two-dimensions are presented to verify the ability of our metric tensor to generate anisotropic mesh with correct concentration and stretching direction.

Key words. Metric tensor, anisotropic mesh, adaptive finite element, node placement and local mesh generation.

1. Introduction

The advantage of anisotropic adaptive mesh refinement method has been amply demonstrated for improving computational efficiency and enhancing the solution accuracy, especially for the problems with anisotropic features. Through adapting the mesh size, shape and orientation, the mesh can be refined both in regions and directions with large errors. The use of anisotropic mesh refinement method involves several key factors: error estimates, determination of metric tensor and anisotropic mesh generation. Beginning with the pioneering work of D'Azevedo[7] and Simpson[27], these techniques have been developed by many researchers[24, 3, 5, 9, 12, 18].

Deriving an efficient and reliable a posteriori error estimator is a difficult task on highly anisotropic mesh. Two requirements must be satisfied for anisotropic error estimators. The error estimator must perform well on anisotropic meshes and should provide the directional information to refine the mesh with large errors. Unfortunately, the classical isotropic a posteriori error estimators can't suit the requirements. For isotropic error estimator, the effectivity index of estimator depends on the mesh aspect ratio which is unbounded for anisotropic mesh. Since the early nineties of last century, many anisotropic a posteriori error estimators have been proposed, for example, the hierarchical a posteriori error estimator[15], the dual weighted residual estimator[11], local problem estimates[2] and so on. In order to specify the refinement direction most of the present error estimators make use of the gradient or Hessian matrix of the solution which are unavailable in numerical computation. To avoid the difficulty, the information of the solution is approximated by the recovery technique such as the Zienkiewicz-Zhu post-processing[34, 35]. It is worth pointing out that although no convergence can be certified in anisotropic

Received by the editors July 16, 2018 and, in revised form, March 3, 2019.
2000 *Mathematics Subject Classification.* 65N30, 65N50.
*Corresponding author.

mesh, numerical results show that the information obtained from recovery technique can be used to guide refinement and coarsening for anisotropic adaptive mesh refinement.

Obtaining the metric from a posteriori error estimator to guide the mesh generation is important for anisotropic mesh generation. Different from isotropic mesh refinement where only the mesh size need to be optimized, for anisotropic mesh refinement method the shape and orientation also need to be optimized. All of them are described by the metric tensor, determined by error estimates. A number of strategies have been developed to deduce metric tensors. The metric tensors are commonly defined as the Hessian matrix of the solution proposed by D'Azevedo[7]. Based on error estimates for polynomial preserving interpolation estimation, Huang[14] developed a general formula for the metric tensor. For the anisotropic elliptic problems Huang et al.[16] verified that high-accuracy finite element solution and superconvergence on the mesh vertices can be obtained by utilizing the inverse of the anisotropic diffusion matrix as the metric tensor for anisotropic mesh generation.

For high-quality anisotropic adaptive mesh generation, three basic approaches exist to achieve mesh refinement: mesh smoothing, anisotropic re-meshing and mesh splitting[24]. For mesh smoothing method, the nodes are relocated at each refinement level to minimize the error estimates. For example, Schneider and Jimack[25] introduced a new anisotropic mesh adaptation strategy in order to modify the node positions of a given (isotropic) mesh such that the a posteriori error estimate is minimized. However how to choose the initial mesh vertex number for the mesh smoothing method is still an open problem. Mesh splitting is a canonical way to refine the mesh for isotropic adaptive mesh refinement method. Whereas the strong anisotropic mesh can't be obtained no matter which kind of splitting strategies are used (the longest edge bisection method or the newest vertex bisection method), since splitting methods limit the aspect ratio for anisotropic mesh. Many researchers[28] show that the anisotropy of the mesh can be increased by pre-defined refinement patterns. Anisotropic re-meshing method requires generating new anisotropic mesh at each refinement level. The mesh with high quality and strong anisotropy can be arrived in fewer steps. For instance the anisotropic centroidal Voronoi tessellation (ACVT) have been developed by Du and Wang[10] for two dimensional anisotropic mesh generation and optimization. There are also a number of computer codes including BL2D[19], BAMG[13], and MMG3D[8] for generating anisotropic meshes which lead to a large number of publications.

In this paper, the focus is on the anisotropic adaptive mesh refinement method based on anisotropic bubble-type local mesh generation (ABLMG) method. The ABLMG-based adaptive mesh generation method proposed in this paper is an anisotropic re-meshing method. Initially, Shimada et al. [26] proposed the bubble packing method (BPM) based on the fact that the force-balance configuration of bubbles forms a well designed node set. The BPM can be used to generate the anisotropic mesh such as the parametric surface mesh [32] and polygonal surface mesh [29], in which the circle bubbles are replaced by ellipse bubbles. In order to avoid using mesh topology, a pure node placement method by bubble simulation (NPBS) was proposed by Liu et al.[20] in which the adjacent list structure is set up to reduce the time of calculating interaction forces. For the node set with high quality generated by NPBS, a fast bubble-type local mesh generation method (BLMG) is presented in [6] and the anisotropic version (ABLMG) is presented in