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A POSTERIORI ERROR ANALYSIS OF AN AUGMENTED DUAL-MIXED METHOD IN LINEAR ELASTICITY WITH MIXED BOUNDARY CONDITIONS

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Abstract. We consider an augmented mixed finite element method for the equations of plane linear elasticity with mixed boundary conditions. The method provides simultaneous approximations of the displacements, the stress tensor and the rotation. We develop an a posteriori error analysis based on the Ritz projection of the error and the use of an appropriate auxiliary function, and derive fully local reliable a posteriori error estimates that are locally efficient up to the elements that touch the Neumann boundary. We provide numerical experiments that illustrate the performance of the corresponding adaptive algorithm and support its use in practice.

Key words. a posteriori error estimates, mixed finite element, augmented formulation, stabilization, linear elasticity, Ritz projection.

1. Introduction

In this work, we consider the problem of plane linear elasticity with mixed boundary conditions. Typically, mixed finite element methods are used in linear elasticity to avoid the effects of locking while approximating additional unknowns of physical interest directly. It is well known that stable mixed finite elements for the linear elasticity problem involve many degrees of freedom. The application of stabilization techniques, such as augmented formulations, allows to use simpler finite element subspaces, including convenient equal-order interpolations that are generally unstable within the mixed approach.

In this framework, we consider the stabilized mixed finite element method presented in [7] for the problem of linear elasticity in the plane with homogeneous Dirichlet and non-homogeneous Neumann mixed boundary conditions. The approach in [7] relies on the mixed method of Hellinger and Reissner, that is enriched with suitable residual terms arising from the equilibrium equation, the constitutive law and the relation that defines the rotation in terms of the displacement. This approach leads to a well-posed, locking-free Galerkin scheme for any choice of finite element subspaces when homogeneous Dirichlet boundary conditions are prescribed. The method was successfully extended to the case of non-homogeneous Dirichlet boundary conditions in [8] (the three-dimensional version can be found in [9]).

In the case of mixed boundary conditions, which is the most usual in practice, the Neumann boundary condition is imposed weakly in [7], through the use of a Lagrange multiplier that can be interpreted as the trace of the displacement on the Neumann boundary. The resulting variational formulation has a saddle point structure. Hence, the analysis of the discrete scheme has to be done for any specific choice of finite element subspaces. In particular, it is possible to use Raviart-Thomas elements of the lowest order to approximate the stress tensor, continuous piecewise linear elements to approximate the displacement and piecewise constants

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to approximate the rotation; the Lagrange multiplier on the Neumann boundary can be approximated by continuous piecewise linear elements on a suitable partition of that boundary, as we will see later. We should mention that an extension of the method proposed in [8, 9] to the case of mixed boundary conditions where the Neumann boundary condition is imposed in a strong sense has been analyzed recently in [11]. This extension leads to an augmented variational formulation with a coercive bilinear form and the corresponding Galerkin scheme is well-posed and free of locking for any choice of finite element subspaces.

Concerning the a posteriori error analysis of the augmented scheme presented in [7], an a posteriori error estimator of residual type was proposed in [1] in the case of pure homogeneous Dirichlet boundary conditions. That analysis was extended to the cases of pure non-homogeneous Dirichlet boundary conditions and mixed boundary conditions with non-homogeneous Neumann data in [2]. All these a posteriori error estimators are reliable and efficient, and involve the computation of at least eleven residuals per element. Moreover, they include normal and tangential jumps, and its extension to the three-dimensional case does not seem attractive. Recently, simpler a posteriori error estimators were introduced in [3] for the augmented schemes introduced in [7, 8, 9] in the case of boundary conditions of Dirichlet type. In the case of homogeneous boundary conditions, the new a posteriori error estimator introduced in [3] is reliable, locally efficient and only requires the computation of four residuals per element. Moreover, it is valid for two and three dimensional problems and for any finite element subspaces. When non-homogeneous boundary conditions are imposed, two new reliable a posteriori error estimators, one valid in 2D and 3D, and a second one that is only valid in 2D are proposed in [3]. The latter is locally efficient in the elements that do not touch the boundary and requires the computation of four residuals per element in the interior triangles, five residuals per element in the triangles with exactly one node on the boundary and six residuals per element in the triangles with a side on the boundary. Neither of these a posteriori error estimators require the computation of normal nor tangential jumps, which makes them easy to implement.

Our aim in this paper is to extend the analysis from [3] to the augmented dual-mixed method introduced in [7] in the case of mixed boundary conditions. With that purpose, we develop an a posteriori error analysis based on the Ritz projection of the error and obtain an a posteriori error estimator that is reliable and efficient, but that contains a non-local term. We then introduce an auxiliary function and derive fully local a posteriori error estimates that are reliable and locally efficient up to those elements that touch the Neumann boundary (see Theorem 5 below). As compared with the a posteriori error estimator introduced in [2] in the case of mixed boundary conditions, the a posteriori error estimates presented here are less expensive and easier to implement. Numerical experiments support the use of the new a posteriori error estimates in practice.

The rest of the paper is organized as follows. In Section 2, we recall the augmented variational formulation proposed in [7] for the linear elasticity problem in the plane with mixed boundary conditions and describe simple finite element subspaces that lead to a well-posed, locking-free Galerkin scheme. In Section 3, we develop the a posteriori error analysis and propose the new a posteriori error estimators. Finally, in Section 4 we provide several numerical experiments that support the use of the new a posteriori error estimates in practice.

We end this section with some notations to be used throughout the paper. Given a Hilbert space H, we denote by H^2 (resp., $H^{2\times 2}$) the space of vectors (resp., square