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USING p-REFINEMENT TO INCREASE BOUNDARY DERIVATIVE CONVERGENCE RATES

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Abstract. Many important physical problems, such as fluid structure interaction or conjugate heat transfer, require numerical methods that compute boundary derivatives or fluxes to high accuracy. This paper proposes a novel approach to calculating accurate approximations of boundary derivatives of elliptic problems. We describe a new continuous finite element method based on p-refinement of cells adjacent to the boundary that increases the local degree of the approximation. We prove that the order of the approximation on the p-refined cells is, in 1D, determined by the rate of convergence at the mesh vertex connecting the higher and lower degree cells and that this approach can be extended, in a restricted setting, to 2D problems. The proven convergence rates are numerically verified by a series of experiments in both 1D and 2D. Finally, we demonstrate, with additional numerical experiments, that the p-refinement method works in more general geometries.

Key words. Finite elements, superconvergence, elliptic equations, numerical analysis, scientific computing.

1. Introduction

Simulation of many important physical problems, such as fluid structure interaction and conjugate heat transfer, requires numerical methods that compute boundary derivatives or fluxes to high accuracy. In some circumstances the only desired result of a calculation is a quantity derived from the boundary derivatives, such as a flux or stress: this problem has long been recognized as one of importance, and a variety of methods (see, e.g., [12, 16, 28]) have been proposed that allow reconstruction of an accurate boundary flux from less accurate interior data. Accurate boundary derivatives are also required for some numerical boundary conditions. For example, in [23] the authors presented a new discrete boundary condition for a fluid-structure interaction problem based on matching accelerations, instead of velocities, and obtained a traction boundary condition involving second derivatives of the fluid velocity. This boundary condition was the key ingredient in a new partitioned algorithm that was high-order, partitioned, and stable without subiterations. While standard in the finite difference community (see, e.g., [9, 23]) these equations, usually called *compatibility boundary conditions*, are not commonly used in finite element methods, though they have appeared in some recent work [8].

A variety of algorithms have been proposed for calculating higher order derivative values from lower order data calculated by a finite element method (see, e.g., [11, 17, 28, 29, 30]): most of these algorithms rely on *data post-processing*, where one uses least squares or other fitting procedure to fit a higher-degree polynomial through known superconvergence points, as discussed in [3]. Another class of methods relies on the application of high-order finite difference stencils to data derived on either a uniform or quasi-uniform grid [19]. A common feature of several postprocessing techniques is that they require a grid satisfying some smoothness condition: without

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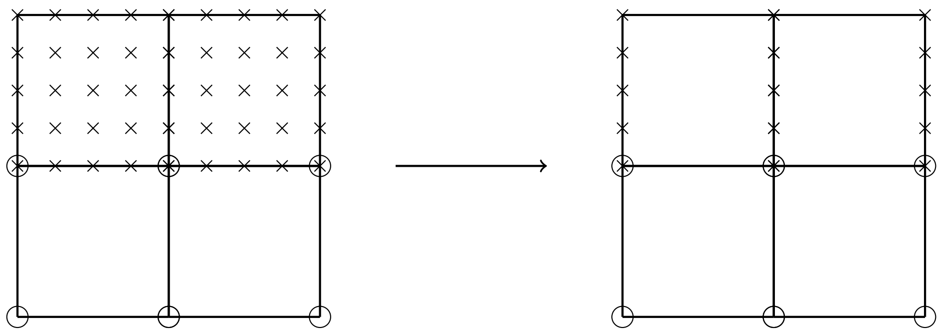

FIGURE 1. Two different implementations of p-refinement for boundary cells adjacent to interior bilinear cells, where the finite element spaces are chosen as nodal interpolants. The diagram on the left is of Q^1 elements adjacent to Q^4 elements: the degrees of freedom with support points along the two common faces would ordinarily be constrained in a way that makes the solution continuous. The scheme proposed in Section 3 uses a similar procedure to constrain *all* such nonnormal degrees of freedom on each boundary cell, effectively reducing the local approximation space to tensor products of $P^1(x)$ and $P^4(y)$. Since the degree of the approximation in the normal direction determines the derivative convergence rates, one could obtain the same effect by adding degrees of freedom corresponding to normal derivatives on the boundary instead of doing Lagrange p-refinement.

such a condition, the error in the solution may be dominated by pollution error from grid irregularities; see Chapter 4 of [3] for additional information on the impact of grid regularity. In particular, of the three most common versions of the finite element method (h-refinement based, p-refinement based, and hp-refinement based) these postprocessing methods are almost always based on estimates from the h-refinement version.

This paper proposes a novel alternative to current techniques. We present a boundary cell *p-refinement* (i.e., locally increasing the degree of the approximation space) strategy to improve the accuracy of boundary derivatives instead of postprocessing the solution. The numerical experiments in Section 4 use Lagrange p-refinement to increase the local approximation degree: a possible alternative to this is to add degrees of freedom corresponding to normal derivatives on the boundary. This p-refinement results in higher rates of convergence in the normal derivatives along the boundary. The theoretical results are based on the two-dimensional linear convection-diffusion-reaction problem

$$-\Delta u + \vec{b} \cdot \nabla u + cu = f \tag{1}$$

with homogeneous Dirichlet boundary conditions in y, periodic boundary conditions in x, normalized viscosity, constant advection velocity $\vec{b}$, constant reaction rate $c > 0$ (which is the standard well-posedness assumption; see Lemma 5.1 in [26] or Chapters 3 and 4 of [24] for further discussion and justification), and forcing f.