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# DISTRIBUTED LAGRANGE MULTIPLIER-FICTITIOUS DOMAIN FINITE ELEMENT METHOD FOR STOKES INTERFACE PROBLEMS

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**Abstract.** In this paper, the distributed Lagrange multiplier-fictitious domain (DLM/FD) finite element method is studied for a type of steady state Stokes interface problems with jump coefficients, and its well-posedness, stability and optimal convergence properties are analyzed by proving an *inf-sup* condition for a nested saddle-point problem that is induced by both Stokes equations and DLM/FD method in regard to Stokes variables (velocity and pressure) and Lagrange multipliers. Numerical experiments validate the obtained convergence theorem of DLM/FD finite element method for Stokes interface problems with respect to different jump ratios.

**Key words.** Stokes interface problems, jump coefficients, distributed Lagrange multiplier, fictitious domain method, mixed finite element, well-posedness, error estimates.

## 1. Introduction

Physicists and engineers use two phase flows to model a wide range of natural phenomena. One such application is a Stokes flow with jump in the viscosity across an interface. This can lead to kinks in the velocity field or discontinuities in pressure at the interface. Using standard finite element methods to capture the true nature of the solution near the interface presents a number of challenges. One method for handling an interface problem is to create a mesh which conforms to the interface [19]. If the interface changes with time, the mesh must be redrawn at each time step to conform with the moving interface. The Arbitrary Lagrangian-Eulerian method [15, 8] is able to adapt the mesh to small movements or changes in the interface, but larger movements or deformations of the interface require that the mesh be redrawn for the whole domain or part of the domain. However, it could be very complicated, time consuming, and less accurate. Furthermore, the transfer of solutions from the degenerated mesh to the new mesh may introduce artificial diffusions, causing loss of accuracy.

Therefore, methods which allow each sub-domain to extend beyond the interface have become increasingly popular. The extended finite element (XFEM) [9, 18] method allows for a mesh which does not conform to the interface. In the XFEM, the interface passes through elements. However, if the ratio of the areas or volumes on either side of the interface becomes too large in any one element, the system can become ill-conditioned and lead to breakdown problems with iterative linear solvers. More recently, the cut finite element method [14, 17] was developed to overcome this problem with the XFEM.

Fictitious domain methods were first developed to handle partial differential equations in a complex geometry [16, 24, 25, 20, 21, 23]. The idea behind the fictitious domain method is to extend the problem from a complicated domain to

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a larger, simpler domain where the problem can be solved more efficiently. When finite element method is used, this allows for a simpler, more regular mesh. In addition, the domain in which both the fluid and fictitious fluid are filled and its mesh are time independent even when the original fluid domain is time dependent.

Lagrange multipliers defined on the actual boundary were later added to implement the genuine boundary conditions [12, 13]. These boundary supported Lagrange multiplier based methods that were first developed for linear elliptic problems were later adapted to non-linear time dependent problems such as the Navier-Stokes equations. The distributed Lagrange multiplier fictitious domain (DLM/FD) method was developed for flows around rigid bodies and the particulate flow problem [10, 11]. In [28] the DLM/FD method is applied to fluid/flexible-body interactions, and a decoupled scheme is developed to solve for the fluid, solid, and Lagrange multiplier terms separately. While some interesting numerical results are provided in those papers, no theoretical analysis is given for DLM/FD finite element method until recently, this method is analyzed for the elliptic interface problem [1, 5] and the parabolic interface problem with a moving interface [27], where, the well-posedness and convergence theorems of the DLM/FD method are proved for those type of interface problems.

Most recently, the DLM/FD method is applied to fluid structure interaction (FSI) problems involving an incompressible viscous-hyperelastic solid [4]. Its stationary case which is defined at each discrete time step is analyzed and an optimal convergence theorem is obtained. In this setting the solid material exhibits both solid and fluid-like properties, with its Cauchy stress tensor given by $\boldsymbol{\sigma}_s = \boldsymbol{\sigma}_s^f + \boldsymbol{\sigma}_s^s$, the sum of a fluid-like part and an elastic part. Thus the influence of fictitious fluid to the structure equation is completely removed from the DLM/FD formulation. Such a specific choice of structure material significantly simplifies the DLM/FD formulation, which is, however, not for a general case of FSI problems.

In this paper, we will take DLM/FD method and apply it to the Stokes interface problem first, where we have a domain $\Omega$ which is divided into two sub-domains $\Omega_1$ and $\Omega_2$ with a jump in the viscosity term across the interface $\Gamma$. The idea behind the fictitious domain method is to create two non-matching meshes. A background mesh is created over the entire domain $\Omega$. A second mesh is generated in the subdomain $\Omega_2$ on one side of the interface. The mesh for $\Omega_2$ then sits on top of the background mesh. The fluid in $\Omega_1$ is extended into the entire domain $\Omega$ and then a distributed Lagrange multiplier (physically a pseudo body force) is used to enforce the fictitious fluid to satisfy the constraint of the structure motion in $\Omega_2$.

Note that the DLM/FD method for the Stokes interface problem results one nested saddle-point problem, where the inner saddle-point problem arises from Stokes equations, and the outer saddle-point problem is induced by the DLM/FD method itself regarding Lagrange multiplier and Stokes variables. It is challenging how to accurately analyze the well-posedness, stability and optimal convergence properties for such a nested saddle-point structure based on the Babuška–Brezzi's theory [2, 6]. In this paper, we will develop a new analysis tool to tackle this problem. In fact, when analyzing the DLM/FD method for the Stokes interface problem, we require that the viscosity of the fluid in $\Omega_1$ be less than the viscosity of the fictitious fluid in $\Omega_2$. We are then able to prove that our DLM/FD formulation is well posed at both the continuous and discrete levels, and derive an optimal error