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ANALYSIS OF A SPECIAL IMMERSED FINITE VOLUME METHOD FOR ELLIPTIC INTERFACE PROBLEMS

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Abstract. In this paper, we analyze a special immersed finite volume method that is different from the classic immersed finite volume method by choosing special control volumes near the interface. Using the elementwise stiffness matrix analysis technique and the H^1-norm-equivalence between the immersed finite element space and the standard finite element space, we prove that the special finite volume method is uniformly stable independent of the location of the interface. Based on the stability, we show that our scheme converges with the optimal order $\mathcal{O}(h)$ in the H^1 space and the order $\mathcal{O}(h^{3/2})$ in the L^2 space. Numerically, we observe that our method converges with the optimal convergence rate $\mathcal{O}(h)$ under the H^1 norm and with the the optimal convergence rate $\mathcal{O}(h^2)$ under the L^2 norm all the way even with very small mesh size h, while the classic immersed finite element method is not able to maintain the optimal convergence rates (with diminished rate up to $\mathcal{O}(h^{0.82})$ for the H^1 norm error and diminished rate up to $\mathcal{O}(h^{1.1})$ for L^2-norm error), when h is getting small, as illustrated in Tables 4 and 5 of [35].

Key words. Immersed finite volume method, stability, optimal convergence rates, immersed finite element method.

1. Introduction

Interface problem is a kind of problem whose domain is typically separated by some curves or surfaces. It appears in many physical applications, such as fluid dynamics [25, 26, 9, 13, 14, 23], electromagnetic problems [4, 36, 38, 40] and materials sciences problems [31]. Many numerical methods have been applied to solve the interface problems. For standard finite element (FE) method on regular meshes, some large errors would probably happen near the interface. To eliminate these errors, the body-fitting FE method has been developed [2, 20, 53, 3, 11]. However, both of them bring the mesh generation complicated, which is particularly unadapted to the moving interface problem. To overcome the drawback, developed are a bunch of methods using a special sort of local basis functions to handle the interface based on Cartesian mesh. For example, the extended finite element method (X-FEM) [57, 51], aiming at extending the solution space via the discontinuous functions; the immersed finite element (IFE) method carrying the idea of taking the special local basis functions on an interface element [34, 10, 12, 37, 21, 32, 55, 56, 52, 8]; the enriched finite element method, based on the IFE technique with the addition of new nodal basis functions [49]. Besides, the Peskin's immersed boundary method (IBM) [41, 42], usually applied to simulate the fluid-structure interactions where the fluid is represented by an Eulerian coordinate while the structure is expressed with a Lagrangian coordinate, takes advantage of the delta function to handle the discontinuous coefficients and the jump conditions.

Unsatisfactorily however, for X-FEM, usually some penalized (or stabilized) terms have to be included in the variational formula. With regard to the IFE method, the L^2-norm convergence rate will degenerates when the mesh becomes very finer. When it comes to the enriched FE method, more degrees of freedom

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have to be imposed near the interface. As for the IBM, it is the first order accurate method and smears the solution near the interface. In this paper, we choose the linear immersed finite element space as the trial space of finite volume (FV) method which enables us to use a uniform Cartesian mesh to solve the elliptic interface problems. There are two advantages in doing this: first, it has no need to consider mesh regeneration in a moving interface; second, many efficient solvers and numerical methods are designed for this Cartesian mesh, for example, fast Poisson solvers [1] and Particle In Cell method for plasma particle Simulation [28, 29, 50, 36].

The FV method has been widely used in numerical solutions of the partial differential equations, a key feature of which is preserving the local conservation laws. At present, the FV method is very popular in computational fluid dynamics, engineering computing, and applied in solving the hyperbolic problems [18, 24, 39, 43, 44, 45, 46, 47, 48, 5, 6]. Combining the advantage of IFE method with the local conservation property of FV mehtod, a linear immersed finite volume (IFV) method is consequently proposed [15, 16, 17, 22].

In this paper, we present a special immersed finite volume (SIFV) method on triangular mesh for solving the following elliptical interface problem. Based on the idea of the literature [54], we demonstrate the stability of the SIFV method by using the elementwise stiffness matrix analysis technique, which differs from the way presented in [16]. We construct modified control volumes for the interface element whereby the proof on stability of the SIFV method is divided into two parts. The first part is about the stability of the interface element, while the second part concerns the stability of the non-interface element. In what follows, we will mainly focus on the former, since the latter is quite natural. We adjust the position of the control vertices with parameters in order to stabilize the interface element. Afterwards, we analyze the convergence of the SIFV method. The differences of our method from the classic IFV [16] and classic IFE method are that not only it guarantees the uniform stability of our scheme in theory, but also numerically makes sure of that the convergence order in the L^2 norm is nondecreasing.

The rest of this paper is organized as follows. In Section 2, we present a special IFV scheme for elliptic equations on the triangular mesh. In Section 3, we prove the stability. In Section 4, we make a convergence analysis. In Section 5, we provide some numerical examples to verify our theories.

We close the section by some standard notations for broken Sobolev space and their associated norms. For any integer $k \geq 0$, we let:

$$\tilde{W}^{k,p}(\Omega) = \{u : u|_{\Omega^s} \in W^{k,p}(\Omega^s), s = -, +\},$$

and define the norm of $\tilde{W}^{k,p}(\Omega)$ as:

$$\|u\|^2_{k,p,\Omega} = \|u\|^2_{k,p,\Omega^-} + \|u\|_{k,p,\Omega^+},$$

and semi-norm as:

$$|u|^2_{k,p,\Omega} = |u|^2_{k,p,\Omega^-} + |u|_{k,p,\Omega^+}.$$

For convenience, we let C denote a generic positive constant which may be different at different occurrences and adopt the following notation. $A \lesssim B$ means $A \leq CB$ for some constants C that are independent of mesh sizes.