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CONVERGENCE ANALYSIS OF FINITE ELEMENT APPROXIMATION FOR 3-D MAGNETO-HEATING COUPLING MODEL

LIXIU WANG, CHANGHUI YAO, AND ZHIMIN ZHANG

Abstract. In this paper, the magneto-heating model is considered, where the nonlinear terms conclude the coupling magnetic diffusivity, the turbulent convection zone, the flow fields, ohmic heat, and α-quench. The highlights of this paper is consist of three parts. Firstly, the solvability of the model is derived from Rothe's method and Arzela-Ascoli theorem after setting up the decoupled semi-discrete system. Secondly, the well-posedness for the full-discrete scheme is arrived and the convergence order $O(h^{\min\{s,m\}} + \tau)$ is obtained, respectively, where the approximation scheme is based on backward Euler discretization in time and Nédélec-Lagrangian finite elements in space. At last, a numerical experiment demonstrates the expected convergence.

Key words. Magneto-heating model, finite element methods, nonlinear, solvability, convergent analysis.

1. Introduction

The phenomenon of magneto-heating has been achieved the main point of interest for many researches[19, 20, 30]. In [19], a magneto-heating model was established and the authors verified the well-posedness of the weak formulation by using the so-called regularity technique. In [24], the authors developed a mathematical model for magnetohydrodynamic flow of biofluids. The main objective was to explore the developmental performance of peristaltic transport with different zeta potentials in conjunction with magnetohydrodynamics and electrodynamics. In [13], the authors were committed to studying the convection flow of an electrically conducting and viscous incompressible fluids through isothermal vertical surfaces with Joule heating, when there exists a uniform transverse magnetic field fixed relative to the surface. Bermúdez and his cooperators studied the coupling of the equations of steady-state magnetohydrodynamics with the power equation when the buoyancy effect is considered in [3]. They showed two models and proved the existence of weak solutions. In [6], the authors researched a coupled system of Maxwell's equations with nonlinear heat equation while they employed the Rothe's method to prove the existence of the weak solutions for this coupled system.

There are many methods to prove the existence of solution for nonlinear equation [7, 10, 21, 23, 31, 37]. Rothe's method presents a first good insight into the structure of the solution of the investigated evolution problem. The method introduced by E. Rothe in 1930 [15]. It relies on the discretization in time and some energy estimates [6]. After then it can be further proved that the discrete solution is convergent to the solution of the original problem. Different from some other abstract methods for confirming the truth of existence, Rothe's method has a strong numerical aspect [15].

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The accurate prediction of magneto-heating phenomena is critical, especially to the basic understanding of the physical principles of controlling the electrodynamics and thermal behavior of the materials in these processing systems [17]. For these purposes, to look for a way to solve such a numerical problem is urgently needed, particularly with the strongly nonlinear conditions. Studies on the finite difference methods and finite volume methods had been applied to the magneto-thermal problems [11, 12, 28, 29]. Meanwhile, finite element method is another important approach for simulating these models due to its superior ability in handling problems that involve complex geometries [1, 2, 33]. It is specially powerful for nonlinear models. In [27], the author studied a nonlinear eddy current model and designed a nonlinear time semi-discrete numerical scheme. Then the Minty-Browder Theorem and a generalization of the div-curl lemma from the steady-state to the transient case were adopted to prove the convergence. As a result, the error estimates were achieved in time. In [5], for stellar magnetic activities, the authors proved the well-posedness of the dynamo system governed by a set of nonlinear PDEs with discontinuous physical coefficients in spherical geometry. Furthermore, they presented a full-discrete finite element approximation to the dynamo system and explored its convergence and stability. In [16], the main purpose was to prove an improved error estimate with $O(\tau + h^{\min\{1,\alpha\}})(\alpha > 0)$ for both time and space discretization than that in [9] for Maxwell's equations with a power-law nonlinear conductivity.

In this paper, compared with models mentioned above, the most significant differences of our model which is proposed in [34] can be summed into three points:

- The model is coupled with the turbulent convection zone and the flow fields.
- The nonlinear term concludes α-quench [5, 25].
- The coefficient of magnetic diffusion is temperature-dependent.

In order to get the existence of the weak solutions, we employ the Rothe's method. Firstly, the monotone theory is utilized to verify the unique solutions of time-discrete weak formulations. Then, by using the weak convergence theorem and Arzela-Ascoli theorem, we obtain that the time-discrete solutions of the magneto-heating coupling model converge to the solutions of the weak formulations. Next, we set up the full-discrete decoupled schemes by backward Euler discretization in time and Nédélec-Lagrangian elements in space. Furthermore, after the preparatory work, we obtain the convergence with the rates $O(h^{\min\{s,m\}} + \tau)$, where an a-prior L^∞ assumption of numerical solution is derived. At last, a simple numerical example is designed.

An outline of this paper is as follows. In section 2, we present the detailed information for the model and denotes some notations which will be used frequently in the rest of the paper. In Section 3, we employ time discretization based on Rothe's method to verify the solvability of the weak solutions for the problem (see Theorem 3.1). In Section 4, we construct the full-discrete scheme. Then based on interpolation theorem and the approximation properties between interpolations and finite element solutions, we obtain the error estimates (see Theorem 4.1), where an a-prior L^∞ assumption has to be inserted since the numerical scheme is the explicit decoupled. In Section 5, a numerical experiment is presented to verify theoretical results. Finally, some concluding remarks are given in the last section.