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A CONFORMING DISCONTINUOUS GALERKIN FINITE ELEMENT METHOD

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Abstract. A new finite element method with discontinuous approximation is introduced for solving second order elliptic problem. Since this method combines the features of both conforming finite element method and discontinuous Galerkin (DG) method, we call it conforming DG method. While using DG finite element space, this conforming DG method maintains the features of the conforming finite element method such as simple formulation and strong enforcement of boundary condition. Therefore, this finite element method has the flexibility of using discontinuous approximation and simplicity in formulation of the conforming finite element method. Error estimates of optimal order are established for the corresponding discontinuous finite element approximation in both a discrete H^1 norm and the L^2 norm. Numerical results are presented to confirm the theory.

Key words. Weak Galerkin, discontinuous Galerkin, finite element methods, second order elliptic problem.

1. Introduction

For the sake of clear presentation, we consider Poisson equation with Dirichlet boundary condition in two dimension as our model problem. This conforming DG method can be extended to solve other elliptic problems. The Poisson problem seeks an unknown function u satisfying

$$-\Delta u = f, \quad \text{in } \Omega, \tag{1}$$

$$u = g, \quad \text{on } \partial\Omega, \tag{2}$$

where Ω is a polytopal domain in $\mathbb{R}^2$.

Researchers started to use discontinuous approximation in finite element procedure in the early 1970s [2, 7, 12, 17]. Local discontinuous Galerkin methods were introduced in [6]. Then a paper [1] in 2002 provides a unified analysis of discontinuous Galerkin (DG) finite element methods for Poisson equation. Since then, many new finite element methods with discontinuous approximations have been developed such as hybridizable discontinuous Galerkin (HDG) method [5], mimetic finite differences method [10], hybrid high-order (HHO) method [11], virtual element (VE) method [13], weak Galerkin (WG) method [14] and references therein.

The weak form of the problem (1)-(2) is given as follows: find $u \in H^1(\Omega)$ such that $u = g$ on $\partial\Omega$ and

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega). \tag{3}$$

The conforming finite element method for the problem (1)-(2) keeps the same simple form as in (3). However, when discontinuous approximation is used, finite element formulations tend to be more complex than (3) to ensure connection of discontinuous function across element boundary. For example, the following is the formulation for the symmetric interior penalty discontinuous Galerkin (IPDG)

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method for the Poisson equation (1) with homogeneous boundary condition: find $u_h \in V_h$ such that for all $v_h \in V_h$,

$$\sum_{T\in\mathcal{T}_h} (\nabla u_h, \nabla v_h)_T - \sum_{e\in\mathcal{E}_h} \int_e \Big(\{\nabla u_h\}[v_h] + \{\nabla v_h\}[u_h] - \alpha h_e^{-1}[u_h][v_h] \Big) = (f, v_h),$$

where α is called a penalty parameter that needs to be tuned.

A first order weakly over-penalized symmetric interior penalty method is proposed in [3] aiming for simplifying the above IPDG formulation by eliminating the two nonsymmetric middle terms: find $u_h \in V_h$ such that for all $v_h \in V_h$,

$$\sum_{T\in\mathcal{T}_h} (\nabla u_h, \nabla v_h)_T + \alpha \sum_{e\in\mathcal{E}_h} h_e^{-3}(\Pi_0[u_h],\ \Pi_0[v_h])_e = (f, v_h),$$

where Π_0 is the L^2 projection to the constant space and α is a positive number. The price paid for a simpler formulation is a worse condition number for the resulting system of linear equations.

In this paper, we propose a new conforming DG method using the same finite element space used in the IPDG method for any polynomial degree $k \geq 1$ but having a simple symmetric and positive definite system: find $u_h \in V_h$ satisfying $u_h = I_h g$ on $\partial\Omega$ and

$$(\nabla_w u_h, \nabla_w v_h) = (f, v_h) \quad \forall v_h \in V_h^0, \tag{4}$$

where ∇_w is called weak gradient introduced in the weak Galerkin finite element method [14, 15]. It follows from (4) that the conforming DG method can be obtained from the conforming formulation simply by replacing ∇ by ∇_w and enforcing the boundary condition strongly. The simplicity of the conforming DG formulation will ease the complexity for implementation of DG methods. The computation of weak gradient $\nabla_w v$ is totally local. Optimal convergence rates for the conforming DG approximation are obtained in a discrete H^1 norm and in the L^2 norm. This new conforming DG method is tested numerically for $k = 1, 2, 3, 4$ and 5, and the results confirm the theory.

2. Finite Element Method

In this section, we will introduce the conforming DG method. For any given polygon $D \subseteq \Omega$, we use the standard definition of Sobolev spaces $H^s(D)$ with $s \geq 0$. The associated inner product, norm, and semi-norms in $H^s(D)$ are denoted by $(\cdot,\cdot)_{s,D}$, $\|\cdot\|_{s,D}$, and $|\cdot|_{s,D}$, respectively. When $s = 0$, $H^0(D)$ coincides with the space of square integrable functions $L^2(D)$. In this case, the subscript s is suppressed from the notation of norm, semi-norm, and inner products. Furthermore, the subscript D is also suppressed when $D = \Omega$.

Let $\mathcal{T}_h$ be a triangulation of the domain Ω with mesh size h that consists of triangles. Denote by $\mathcal{E}_h$ the set of all edges in $\mathcal{T}_h$, and let $\mathcal{E}_h^0 = \mathcal{E}_h \backslash \partial\Omega$ be the set of all interior edges.

We define the average and the jump on edges for a scalar-valued function v. For an interior edge $e \in \mathcal{E}_h^0$, let T_1 and T_2 be two triangles sharing e. Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be the two unit outward normal vectors on e, associated with T_1 and T_2, respectively. Define the average $\{\cdot\}$ and the jump $[\cdot]$ on e by

$$\{v\} = \frac{1}{2}(v|_{T_1} + v|_{T_2}) \quad \text{and} \quad [v] = v|_{T_1}\mathbf{n}_1 + v|_{T_2}\mathbf{n}_2, \tag{5}$$

respectively. If e is a boundary edge, then

$$\{v\} = v, \quad [v] = v\mathbf{n}. \tag{6}$$