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# 3D $B_2$ MODEL FOR RADIATIVE TRANSFER EQUATION

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**Abstract.** We proposed a 3D $B_2$ model for the radiative transfer equation. The model is an extension of the 1D $B_2$ model for the slab geometry. The 1D $B_2$ model is an approximation to the 2nd order maximum entropy ($M_2$) closure and has been proved to be globally hyperbolic. In 3D space, we are basically following the method for the slab geometry case to approximate the $M_2$ closure by $B_2$ ansatz. Same as the $M_2$ closure, the ansatz of the new 3D $B_2$ model has the capacity to capture both isotropic solutions and strongly peaked solutions. And beyond the $M_2$ closure, the new model has fluxes in closed-form such that it is applicable to practical numerical simulations. The rotational invariance, realizability, and hyperbolicity of the new model are carefully studied.

**Key words.** Radiative transfer, moment model, maximum entropy closure.

## 1. Introduction

The radiative transfer equations describe the transportation of photons in a medium [22, 20]. They are *kinetic equations*, and the unknown is the specific intensity of photons. The specific intensity is a function of time, spatial coordinates, frequency, and angular variables. There are numerous methods for solving the radiative transfer equations [12, 5, 26, 8, 19]. The moment method is an efficient approach for reducing the computation cost brought about by the high-dimensionality of variables of kinetic equations.

In most applications, the quantities of interest are the few lowest order moments. Therefore moments are proper choices for discretizing the angular variables. In fact, in many applications, people are only concerned with the zeroth order moment and a diffusion equation is often solved to approximate the radiation process [29]. However, the diffusion equation might not be a very accurate approximation when the radiation field is away from equilibrium, therefore more moments are sometimes needed. An essential problem in the moment method is that moment systems are not closed. Closing the system by specifying a constitutive relationship is known as the *moment-closure problem*. One approach towards moment-closure is to recover the angular dependence of the specific intensity from the known moments. The reconstructed specific intensity is called an *ansatz*. Ideally, the ansatz should be non-negative for all moments which can be generated by a non-negative distribution. Also, one would like the system to be hyperbolic since hyperbolicity is necessary for the local well-posedness of Cauchy problem. Other natural requirements include that the ansatz satisfies rotational invariance and reproduces the isotropic distribution at equilibrium. Numerous forms of ansätze have been studied in the literature. For detailed descriptions of standard methods we refer to [22, 18]. Yet, in multi-dimensional cases, the maximum entropy method, referred to as the $M_n$ model, is perhaps the only method known so far to have both

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realizability and global hyperbolicity [7]. However, the flux functions of the maximum entropy method are generally not explicit [1], so numerically computing such models involve solving highly nonlinear and probably ill-conditioned optimization problems frequently. There have been continuous efforts on speeding up the computation process [2, 1, 10]. Recently, there are also attempts in deriving closed-form approximations of the maximum entropy closure in order to avoid the expensive computations. For 1D cases, an approximation to the $M_n$ models using the Kershaw closure is given in [23]. For multi-dimensional cases, a model based on directly approximating the closure relations of the $M_1$ and $M_2$ methods is proposed in [21]. Our work in this paper also aims at constructing closed-form approximations of the maximum entropy model. Like [21], we seek a closed-form approximation to the $M_2$ method in 3D. But unlike [21], we derive our model from an ansatz with some similarity to that of the $M_2$ model.

In a previous study [3], we analyzed the second order extended quadrature method of moments (EQMOM) introduced in [27] which we call the $B_2$ model. In this work, we propose an approximation of the $M_2$ model in 3D space by extending the $B_2$ model studied in [3] to 3D. The reason for this approach is that the $B_2$ ansatz shares the following properties with the $M_2$ ansatz:

(1) it interpolates smoothly between isotropic and strongly peaked distribution functions;
(2) it captures anisotropy in opposite directions.

The $B_2$ closure in [3] is for slab geometries. Preserving rotational invariance when extending it to 3D space is non-trivial. We use the sum of the axisymmetric $B_2$ ansätze in three mutually orthogonal directions as the ansatz for a second order moment model in 3D space. This new model is referred to as the *3D $B_2$ model*. The consistency of known moments requires the three mutually orthogonal directions to be the three eigenvectors of the second-order moment matrix. We point out that there are three free parameters in the ansatz of the 3D $B_2$ model after the consistency of known moments is fulfilled. These parameters are specified as functions of the first-order moments and the eigenvalues of the second-order moment matrix. We prove that the 3D $B_2$ model is rotationally invariant. The region where the model possesses a non-negative ansatz is illustrated, as well as the hyperbolicity region of the model with vanished first-order moment. Though far from perfect, the 3D $B_2$ model shares some important features of the $M_2$ closure. Also, the model has explicit flux functions, making it very convenient for numerical simulations.

The rest of this paper is organized as follows. In Section 2 we recall the basics of moment models, and briefly, introduce the $M_2$ method as well as the $B_2$ model for 1D slab geometry. In Section 3 we propose the 3D $B_2$ model. In Section 4 we analyze its properties. Finally, in Section 5 we summarize and discuss future work.

## 2. Preliminaries

The specific intensity $I(t, \mathbf{r}, \nu, \mathbf{\Omega})$ is governed by the radiative transfer equation

$$\frac{1}{c}\frac{\partial I}{\partial t} + \mathbf{\Omega} \cdot \nabla I = \mathcal{C}(I), \tag{1}$$

where $c$ is the speed of light. The variables in the equation are time $t \in \mathbb{R}^+$, the spatial coordinates $\mathbf{r} = (x, y, z) \in \mathbb{R}^3$, the angular variables $\mathbf{\Omega} = (\Omega_x, \Omega_y, \Omega_z) \in \mathbb{S}^2$, and frequency $\nu \in \mathbb{R}^+$. The right-hand side $\mathcal{C}(I)$ describes the interactions between

[1] With the first order maximum entropy model for the grey equations as the only exception [7].