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# THE PROPERTY OF THE BRANCH OF NONSINGULAR FINITE ELEMENT/FINITE VOLUME SOLUTIONS TO THE STATIONARY NAVIER-STOKES EQUATIONS AND ITS APPLICATION

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**Abstract.** In this paper, a branch of nonsingular solutions of the stationary Navier-Stokes equations are investigated, which are unique on a neighborhood, and mostly isolated without relying on very stringent requirement on the small data. We summarize and develop an equivalent definition of nonsingular solutions of finite element/finite volume methods in the same framework. Furthermore, we establish the equivalent definition of a branch of singular solutions of finite element methods for the coupled Navier-Stokes/Darcy equations.

**Key words.** Stationary Navier-Stokes equations, finite element methods, finite volume methods, the branch of nonsingular solutions, *inf-sup* condition, large data.

## 1. Introduction

The Navier-Stokes equations usually have more than one solution unless the data satisfy the stringent requirement of uniqueness condition of the solutions, which required that the data be small enough in certain norms [17, 33]. However, this uniqueness condition is rarely satisfied in the real world. In many practical examples, the solutions are mostly isolated, and depend continuously on the viscosity. As the viscosity varies along an interval, each solution of the Navier-Stokes equations describes an isolated branch, which means the bifurcation phenomenon is rare. This situation is expressed mathematically by the notion of branches of nonsingular solutions.

Finite element approximations of nonsingular solutions have been investigated in [4, 17, 18], where optimal order error estimates have been obtained for the stationary Navier-Stokes equations with large data. Also, an analysis of the nonsingular finite volume solutions to the Navier-Stokes equations is not direct to establish where the whole system lacks symmetry in the context of a petrov-Galerkin method [2, 6, 7, 8, 14, 15, 16, 19, 25, 34].

For both finite element/finite volume approximations of the stationary Navier-Stokes equations, the original definition of nonsingular solutions is difficult to be applied and developed for the further research in this field. Here, we apply the definition of an isomorphism between two spaces to obtain the equivalent definition of discrete nonsingular solutions to the stationary Navier-Stokes equations and its coupled system.

This paper is organized as follows: In the next section, we introduce notations of a branch of nonsingular solutions to the stationary Navier-Stokes equations. Then, in the third section, the property of a branch of nonsingular finite element solutions to the stationary Navier-Stokes equations is derived. Also, the corresponding property of nonsingular finite volume solutions is investigated in the fourth section. Finally, we investigate the property of a branch of nonsingular finite element

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solutions to the coupled Navier-Stokes/Darcy model with heterogeneous porous medium.

## 2. A branch of nonsingular solutions to the stationary Navier-Stokes equations

Let $\Omega$ be a bounded domain in $\Re^d, d = 2, 3$, assumed that $\Gamma$ is of $C^2$ or if $\Omega$ is a two-dimensional convex polygon. The stationary Navier-Stokes equations are

$$-\Delta u + \lambda \nabla p = \lambda f - \lambda((u \cdot \nabla)u + \frac{1}{2}(\text{div} u)u), \quad \text{in } \Omega, \tag{1}$$

$$\text{div}\, u = 0, \quad \text{in } \Omega, \tag{2}$$

$$u|_\Gamma = 0, \quad \text{on } \Gamma, \tag{3}$$

where $u = u(x)$ represents the velocity vector, $p = p(x)$ the pressure, $f = f(x)$ the prescribed body force, $\lambda = \mu^{-1}$, and $\mu > 0$ the viscosity.

For simplicity, some useful Sobolev spaces can be defined by:

$$X = [H_0^1(\Omega)]^d,\ M = L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q dx = 0 \right\},\ Z = [L^{3/2}(\Omega)]^d,$$

$$\bar{X} = X \times M,\ Y = [H^{-1}(\Omega)]^d,\ V = \{v \in X : \text{div}\, v = 0\},$$

$$H = \{v \in [L^2(\Omega)]^d : \text{div}\, v = 0\},\ D(A) = [H^2(\Omega)]^d \cap V,$$

where the Stokes operator $A : D(A) \to H$ is defined by $A = -P\Delta$ and $P : [L^2(\Omega)]^d \to H$ is the standard $L^2$-orthogonal projection. The spaces $[L^2(\Omega)]^m$, $m = 1, 2$, or 4, are endowed with the $L^2$-scalar product $(\cdot, \cdot)$ and the $L^2$-norm $\|\cdot\|_{L^2}$, as appropriate. In addition, $\|\cdot\|_{L^r},\ 1 \le r \le \infty$, denotes the norm of the space $L^r(\Omega)$. The space $X$ is equipped with the usual scalar product $(\nabla u, \nabla v)$ and the norm $\|u\|_{H^1}$ (or equivalently $\|\nabla u\|_{L^2}$), $u, v \in X$. In particular, define the norm on $\bar{X}$:

$$|||(v, q)||| = (\|\nabla v\|_{L^2}^2 + \lambda^2 \|q\|_{L^2}^2)^{1/2},\ (v, q) \in \bar{X}.$$

In this paper standard definitions are used for the Sobolev spaces $W^{m,r}(\Omega)$ [1], with the norm $\|\cdot\|_{W^{m,r}}$ and the seminorm $|\cdot|_{W^{m,r}}$, $m, r \ge 0$. We will write $H^m(\Omega)$ for $W^{m,2}(\Omega)$ and $\|\cdot\|_{H^m}$ for $\|\cdot\|_{W^{m,2}(\Omega)}$.

First, we consider the linear Stokes equations in order to introduce some mathematical theory of the nonsingular solutions of the stationary Navier-Stokes equations. A linear operator $T : Y \to \bar{X}$ is defined as follows: Given $g \in Y$, the solution of the Stokes problem

$$\begin{aligned} -\Delta v + \lambda \nabla q = g, &\quad \text{in } \Omega, \\ \text{div}\, v = 0, &\quad \text{in } \Omega, \\ v|_\Gamma = 0, &\quad \text{on } \Gamma, \end{aligned} \tag{4}$$

is denoted by $\tilde{v}(\lambda) = (v, \lambda q) = Tg \in \bar{X}$. Furthermore, a $C^2$-mapping $\mathcal{G} : R^+ \times \bar{X} \to Y$ is defined by

$$\mathcal{G}(\lambda, \tilde{v}(\lambda)) = \lambda \left( (v \cdot \nabla)v + \frac{1}{2}(\text{div}\, v)v - f \right)$$

since the term div$u = 0$. Finally, we define

$$F(\lambda, \tilde{v}(\lambda)) = \tilde{v}(\lambda) + T\mathcal{G}(\lambda, \tilde{v}(\lambda)), \quad \lambda \in R^+,\ \tilde{v}(\lambda) \in \bar{X}.$$

In this section, a branch of nonsingular solutions of the stationary Navier-Stokes equations, as introduced in [4, 17, 18, 22, 23], are studied.