INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 17, Number 1, Pages 54–67



# CONFORMING HARMONIC FINITE ELEMENTS ON THE HSIEH-CLOUGH-TOCHER SPLIT OF A TRIANGLE

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**Abstract.** We construct a family of conforming piecewise harmonic finite elements on triangulations. Because the dimension of harmonic polynomial spaces of degree $\leq k$ is much smaller than the one of the full polynomial space, the triangles in the partition must be refined in order to achieve optimal order of approximation power. We use the Hsieh-Clough-Tocher split: the barycenter of each original triangle is connected to its three vertices. Depending on the polynomial degree $k$, the original triangles have some minor restrictions which can be easily fulfilled by small perturbations of some vertices of the original triangulation. The optimal order of convergence is proved for the conforming harmonic finite elements, and confirmed by numerical computations. Numerical comparisons with the standard finite elements are presented, showing advantages and disadvantages of the harmonic finite element method.

**Key words.** Harmonic polynomial, conforming finite element, triangular grid, Hsieh-Clough-Tocher, Laplace equation.

## 1. Introduction

Standard finite element methods use the full space $P_k$ of polynomials of total degree $\leq k$, or its enrichment by the so-called bubble functions, on each element (e.g. triangle or tetrahedron) for solving partial differential equations. That is, to reach the optimal order of approximation, the traditional finite element space must contain the full space $P_k$ locally, cf. [2, 4, 5, 6, 7, 8, 9, 10, 14, 15, 16, 17].

Instead of using the full polynomial space $P_k$, in this work we use the harmonic polynomials $P_{k,\mathrm{harm}}$ to construct conforming finite elements of optimal order for approximating only harmonic solutions. A harmonic polynomial $p$ in two variables is a harmonic function, i.e., $\Delta p = p_{xx} + p_{yy} = 0$. For each $k > 0$, there are only two harmonic polynomials of exact degree $k$ in two variables. They are the real and the imaginary part of the analytic polynomial $z^k = (x + iy)^k$. The dimension of $P_{k,\mathrm{harm}}$ is $2k + 1$ as opposite to $(k + 1)(k + 2)/2$ for the dimension of $P_k$. But the order of convergence of the corresponding finite elements is the same.

We solve the following boundary value problem that has a harmonic solution:

$$(1)\qquad \begin{aligned} -\Delta u &= 0, &&\text{in } \Omega,\\ u &= f, &&\text{on } \partial\Omega, \end{aligned}$$

where $\Omega$ is a bounded polygonal domain in $\mathbb{R}^2$. If the equation (1) has a non-zero function $g$ on the right-hand side, i.e., $-\Delta u = g$, then the Fourier transform method can be used to find a solution $u_1$ such that $-\Delta u_1 = g$ without any boundary condition. Next the problem is reduced to a homogeneous problem with the boundary condition $f$ in (1) replaced by $f - u_1$. The solution to the original problem is given by $u + u_1$.

The projection of $P_{k,\mathrm{harm}}$ on a line in $\mathbb{R}^2$ is the full space of univariate polynomials of degree $\leq k$. Thus, to construct a harmonic finite element, we need $k + 1$ degrees of freedom on each edge in the triangulation in order to have a continuous finite

Received by the editors April 16, 2019.
2000 *Mathematics Subject Classification.* 65N30, 65N15.

element space (conforming finite element). This would lead at least $3k$ degrees of freedom in each triangle, see Fig. 1. But the dimension of $P_{k,\text{harm}}$ is only $2k+1$. Thus, to construct 2D conforming harmonic finite elements on a triangulation, one has to split the original triangles. In this work, we construct $P_{k,\text{harm}}$ conforming finite elements on Hsieh-Clough-Tocher (H-C-T) refinements of triangulations. An H-C-T refinement is obtained by connecting the barycenter of each triangle to its three vertices. Thus, each macro-triangle in the original triangulation is split into three subtriangles, and we have to work with three harmonic polynomials on one H-C-T macro-triangle. There are $3(2k+1) = 6k+3$ polynomial coefficients to be determined. For the continuity along three internal edges, we impose $3(k+1)-1 = 3k+2$ linear equations. By specifying the nodal values on the boundary and at the barycenter we obtain $3k+1$ equations. The total number of equations, $3k+2+3k+1$, is equal to the number of polynomial coefficients to be determined.

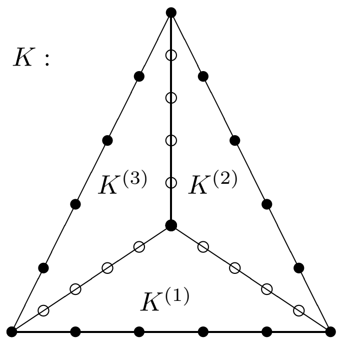


FIGURE 1. In $K$, $\bullet$ denotes a degree of freedom of $P_{5,\text{harm}}$ finite element; $\circ$ denotes a continuity constraints.

However, these equations may not have (unique) solutions. We show that, in general, this depends on the geometry of macro-triangles. In particular, for $k=2$, there are no geometric constraints. For $k=3$, only isosceles triangles are not allowed. For each $k$ afterward, there is an an additional restriction that a certain polynomial function of the three angles does not vanish. Nevertheless, the prohibited combinations of the angles form a zero measure subset of the domain of the angles. So, in computation, we simply perturb one of the three vertices of a triangle if the computer fails to generate basis functions on this macro-triangle.

We prove a special case of the Bramble-Hilbert lemma [3] for approximating harmonic functions by harmonic polynomials. Using the lemma, we show that the harmonic finite elements converge at the optimal order, when solving (1). In the last section, we numerically test the harmonic finite elements of degree 2 to 6, confirming the theoretical results. In addition, numerical comparisons with the standard finite elements are presented. In an earlier work [12], we constructed a $P_{2,\text{harm}}$ conforming finite element on macro-rectangles, and a $P_{2,\text{harm}}$ nonconforming finite element on general non-refined triangulations.

## 2. Definition of harmonic finite elements

Let $\mathcal{M}_h = \cup_{K\in\mathcal{T}_h} K$ be a quasi-uniform triangulation of size $h$ on the polygonal domain $\Omega \subset \mathbb{R}^2$. Depending on the harmonic polynomial degree $k$, cf. Theorem 2.1 below, we may need to perturb some internal vertices a little to form a new triangulation $\tilde{\mathcal{M}}_h$ in the computation. Each triangle $K$ in $\tilde{\mathcal{M}}_h$ is subdivided into