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THEORETICAL AND NUMERICAL STUDIES ON GLOBAL STABILITY OF TRAVELING WAVES WITH OSCILLATIONS FOR TIME-DELAYED NONLOCAL DISPERSION EQUATIONS*

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Abstract. This paper is concerned with the global stability of non-critical/critical traveling waves with oscillations for time-delayed nonlocal dispersion equations. We first theoretically prove that all traveling waves, especially the critical oscillatory traveling waves, are globally stable in a certain weighted space, where the convergence rates to the non-critical oscillatory traveling waves are time-exponential, and the convergence to the critical oscillatory traveling waves are time-algebraic. Both of the rates are optimal. The approach adopted is the weighted energy method with the fundamental solution theory for time-delayed equations. Secondly, we carry out numerical computations in different cases, which also confirm our theoretical results. Because of oscillations of the solutions and nonlocality of the equation, the numerical results obtained by the regular finite difference scheme are not stable, even worse to be blow-up. In order to overcome these obstacles, we propose a new finite difference scheme by adding artificial viscosities to both sides of the equation, and obtain the desired numerical results.

Key words. Critical traveling waves, time-delay, global stability, nonlocal dispersion equation, oscillations.

1. Introduction

In this paper, we consider the global stability of critical oscillatory traveling waves for a class of nonlocal dispersion equations with time-delay

$$\begin{cases} \dfrac{\partial v}{\partial t} - D(J*v-v) + d(v) = K*b(v(t-r,\cdot)), & x\in\mathbb{R},\ t>0, \\ v(s,x) = v_0(s,x), & x\in\mathbb{R},\ s\in[-r,0], \end{cases} \tag{1}$$

where the initial value satisfies

$$\lim_{x\to\pm\infty} v_0(s,x) = v_\pm, \text{ uniformly in } s\in[-r,0].$$

This model represents the spatial dynamics of a single-species population with age-structure and nonlocal diffusion such as the Australian blowflies population distribution [7, 8]. Here $v(t,x)$ denotes the total mature population of the species, the function $d(v)$ and $b(v)$ are the death and birth rates of the mature population respectively, $J(x)$ and $K(x)$ are non-negative, unit and symmetric kernels, $J(x)$ is the probability distribution of rates of dispersal over distance x. Then $J*v(x)$ is the rate at which individuals are arriving at position x from all other locations, and $v(x) = \int_{\mathbb{R}} J(x,y)v(x)dy$ is the rate at which they are leaving location x to travel to all other sites. Therefore, the expression $D(J*v-v)$ is the nonlocal dispersion due to long range dispersion mechanisms [4, 12], where the coefficient $D>0$ is the spatial diffusion rate.

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The advantages of the nonlocal process governed by integral process over the classical dispersal process modelled by Laplacian lie in the fact that the nonlocal one accounts for interaction between individual in both short and long ranges, while the classical one accounts for only local interactions between the neighbor individuals. Moreover, the nonlocal operator for the initial value problem is not a smoothing operator. Discontinuities in the initial data are retained [4]. And the spatial decay rates of the traveling waves at infinity are different in the local and nonlocal cases [33].

From the classical Nicholson's blow flies model [9] with the birth rate function $b(v) = pve^{-av}$ for $p > 0$ and $a > 0$ and the death rate function $d(v) = \delta v$ for $\delta > 0$, and the Mackey-Glass model [17] with $b(v) = \frac{v}{1+av^q}$ for $a > 0$ and $q > 1$ and $d(v) = dv$ for $d > 0$, throughout this paper, we assume the birth rate function, the death rate function, and the kernels to be:

(H1) There exist two constant equilibria of (1): $v_- = 0$ is unstable and $v_+ > 0$ is stable, namely, $d(0) = b(0) = 0$, $d(v_+) = b(v_+)$, $0 \le d'(0) < b'(0)$ and $d'(v_+) > b'(v_+)$;

(H2) Both $d(s)$ and $b(s)$ are non-negative, C^2-smooth functions with $d'(s) \ge d'(0) \ge 0$, $|b'(s)| \le b'(0)$ for $s \in [0, +\infty)$, but $b(s)$ is non-monotone;

(H3) Both kernels $J(x)$ and $K(x)$ are nonnegative, symmetric and unit,

$$J(x) \ge 0, \quad J(-x) = J(x), \quad \int_{\mathbb{R}} J(x)dx = 1,$$
$$K(x) \ge 0, \quad K(-x) = K(x), \quad \int_{\mathbb{R}} K(x)dx = 1,$$

and satisfy

$$\int_{\mathbb{R}} |x|J(x)e^{-\eta x}dx < \infty \quad \text{and} \quad \int_{\mathbb{R}} |x|K(x)e^{-\eta x}dx < \infty \quad \text{for any} \quad \eta > 0.$$

(H4) The Fourier transform of $J(x)$, denoted by $\hat{J}(\xi)$, satisfies that $\hat{J}(\xi) = 1 - \kappa|\xi|^\alpha + o(|\xi|^\alpha)$ as $\xi \to 0$ with $\alpha \in (0, 2]$ and $\kappa > 0$, and $1 - \hat{J}(\xi) \ge \omega(r)$ for all $|\xi| \ge r$ and any $r > 0$ with some positive function $\omega(r) > 0$.

A traveling wavefront of (1) is a special solution of the form $u(t, x) = \phi(x + ct)$, where c is the wave speed. The existence and uniqueness (up to a shift) of traveling waves for the equation (1) were proved in [10, 37, 38]. The main purpose of this paper is to study the global stability of traveling wavefronts $\phi(x+ct)$ of (1), especially the case of the critical wave $\phi(x + c^*t)$. Here the number c^* is called the critical speed (or the minimum speed) in the sense that a traveling wave exists if $c \ge c^*$, while no traveling wave $\phi(x + c^*t)$ exists if $0 < c < c^*$. Let $\phi(x + ct) = \phi(\xi)$ be any given monotone or non-monotone traveling waves for (1) with wave speed $c \ge c^*$ connecting the two steady equilibria $v_\pm$, namely,

$$\begin{cases} c\phi'(\xi) - D\Big(\displaystyle\int_{\mathbb{R}} J(y)\phi(\xi - y)dy - \phi(\xi)\Big) + d(\phi(\xi)) \\ \qquad = \displaystyle\int_{\mathbb{R}} K(y)b(\phi(\xi - y - cr))dy, \qquad \xi \in \mathbb{R}, \\ \phi(\pm\infty) = v_\pm, \quad \phi(\xi) \ge 0, \qquad \xi \in \mathbb{R}, \end{cases} \tag{2}$$

where $\xi = x + ct$, $' = \frac{\partial}{\partial \xi}$. As summarized in [10], we obtain the following characteristic equation for the pair of (c, λ):

$$c\lambda - D\int_{\mathbb{R}} J(y)e^{-\lambda y}dy + D + d'(0) = b'(0)e^{-\lambda cr}\int_{\mathbb{R}} K(y)e^{-\lambda y}dy. \tag{3}$$