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ANALYSIS OF A SECOND-ORDER DECOUPLED TIME-STEPPING SCHEME FOR TRANSIENT VISCOELASTIC FLOW

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Abstract. In this paper, we propose and analyze a decoupled second order backward difference formula (BDF2) time-stepping algorithm for solving transient viscoelastic fluid flow. The spatial discretization is based on continuous Galerkin finite element approximation for the velocity and pressure, and discontinuous Galerkin finite element approximation for the viscoelastic stress tensor. To obtain a non-iterative decoupled algorithm from the fully discrete nonlinear system, we employ a second order extrapolation in time to the nonlinear terms. The algorithm requires the solution of one Navier-Stokes problem and one constitutive equation per time step. For mesh size h and temporal step size Δt sufficiently small satisfying $\Delta t \leq Ch^{d/4}$, a priori error estimates in terms of Δt and h are derived. Numerical tests are presented that illustrates the accuracy and stability of the algorithm.

Key words. Viscoelasticity, finite element method, discontinuous Galerkin method, decoupled scheme, error estimates, BDF2.

1. Introduction

Time accurate computation of viscoelastic flows are important in many engineering applications involving non-Newtonian fluid mechanics, see [13, 17, 21]. The Oldroyd-B model is one of the simplest constitutive models capable of describing the viscoelastic behavior of flows in which the extra stress tensor is defined by a hyperbolic partial differential equation. The challenges posed by the hyperbolic character of the equation for the extra stress tensor such as spurious oscillations warrants care in discretizing this equation. For the steady state problem, a discontinuous Galerkin (DG) finite element approximation of the constitutive equation was proposed and analyzed in [2]. In [16], a decoupled algorithm was analyzed for efficient implementation of the scheme discussed in [2]. In [20], a Streamline Upwind Petrov Galerkin (SUPG) approximation was employed to discretize the constitutive equation and an error analysis was presented. For the unsteady problem, a DG discretization based approximation for the constitutive equation in inertialess flow was studied in [3]. In [5], a fractional step θ method for time integration, combined with Taylor-Hood finite element and the SUPG spatial discretization is presented. An implicit backward Euler time discretization and continuous piecewise linear finite element in space for three field S-tokes problem is discussed in [1]. In [22], unconditional error estimates of finite element approximation to the viscoelastic flows, with DG discretization for the constitutive equation is discussed. With first order implicit Euler

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temporal discretization and Taylor-Hood finite element approximation for the velocity and pressure, they derived error estimates under the assumption $\Delta t \leq Ch^{3/2}$. In [9], a first order implicit Euler time discretization and SUPG discretization for the constitutive equation was discussed and error estimates were derived under the assumption that $\Delta t, \nu < C h^{d/2}$, where ν is the stabilization parameter of SUPG method. In [8], a Crank-Nicolson time discretization scheme with a DG approximation for the constitutive equation presented and error estimates were derived under the assumption that $\Delta t \leq Ch^{d/4}$.

In this paper, we propose and analyze a partitioned time stepping scheme for the viscoelastic flow model based on second order backward Euler time discretization. A second order in time extrapolation is used to effect a decoupling of the subphysics problems and to have the approximation determined at each time level by the solution of a single linear system. With finite element approximation of the momentum equation and DG method for the constitutive equation, we derive error estimates under the assumption $\Delta t \leq Ch^{d/4}$.

The rest of the paper is organized as follows: In Section 2, we introduce the decoupled second-order backward difference time stepping scheme assuming mixed finite element spatial discretizations for the time dependent viscoelastic flow with constitutive equation stabilized by discontinuous Galerkin (DG) approximation. In §3, we present the error estimates for the fully discrete approximations. In §4, we present numerical results that illustrate the accuracy and efficiency of our algorithm. We close by providing some remarks in §5.

2. The Oldroyd B model and decoupled time-stepping scheme

2.1. The Oldroyd B model. We consider a fluid flow in a bounded domain Ω in $\mathbb{R}^d$, $(d = 2, 3)$ with Lipschitzian boundary Γ. Let p denotes the pressure, $\mathbf{u}$ the velocity, $\mathbb{D}(\mathbf{u}) := \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^t)$ the rate of strain tensor and σ_{tot} the total stress tensor. An Oldroyd's model of differential type with a single relaxation time is obtained by setting $\sigma_{\text{tot}} = -pI + \sigma + \sigma_N$ where σ is the viscoelastic part of the extra stress tensor and $\sigma_N = 2(1-\alpha)\mathbb{D}(\mathbf{u})$ is the Newtonian part, $1 < \alpha \leq 1$. The Oldroyd-B model of viscoelastic flow then is the following

(1)
$$\begin{cases} \partial_t \mathbf{u} - \dfrac{2(1-\alpha)}{Re}\nabla \cdot \mathbb{D}(\mathbf{u}) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \dfrac{1}{Re}\nabla p \; - \; \nabla\cdot\sigma = \mathbf{f} \quad \text{in } \Omega\times(0,T] \\ \qquad\qquad\qquad\qquad\qquad\qquad \nabla\cdot\mathbf{u} \; = \; 0 \quad \text{in } \Omega\times(0,T] \\ \partial_t\sigma + (\mathbf{u}\cdot\nabla)\sigma - \frac{2\alpha}{\lambda}\mathbb{D}(\mathbf{u}) + g_a(\sigma,\nabla\mathbf{u}) + \frac{\sigma}{\lambda} \; = \; 0 \quad \text{in } \Omega\times(0,T] \end{cases}$$

where the function $\mathbf{f}$ is the external force and the function g_a is defined by

$$g_a(\sigma, \nabla\mathbf{u}) := \frac{1-a}{2}(\sigma\nabla\mathbf{u} + (\nabla\mathbf{u})^t\sigma) - \frac{1+a}{2}((\nabla\mathbf{u})\sigma + \sigma(\nabla\mathbf{u})^t),$$