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ON THE LINEAR THERMOELASTICITY WITH TWO POROSITIES: NUMERICAL ASPECTS

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Abstract. In this work we analyze, from the numerical point of view, a dynamic problem involving a thermoelastic rod. Two porosities are considered: the first one is the macro-porosity, connected with the pores of the material, and the other one is the micro-porosity, linked with the fissures of the skeleton. The mechanical problem is written as a set of hyperbolic and parabolic partial differential equations. An existence and uniqueness result and an energy decay property are stated. Then, a fully discrete approximation is introduced using the finite element method and the backward Euler scheme. A discrete stability property and a priori error estimates are proved, from which the linear convergence of the algorithm is derived under suitable additional regularity conditions. Finally, some numerical simulations are presented to show the behaviour of the approximation.

Key words. Thermoelasticity with two porosities, finite elements, a priori error estimates, numerical simulations.

1. Introduction

The study of thermoelastic problems with double porosity has become a topic of increasing interest during the last twenty years. Some possible applications of this kind of models have been found, for instance, in geophysics or in biomechanics (bones) [1, 2, 3, 4, 5]. The main idea of this model is to consider two porosities: the first one is the macro-porosity, which is connected with the pores of the material, meanwhile the second one is related to the fissures in the skeleton. Straughan [4] pointed out that "a good example of this may be seen in the pictures in [6] where they show a pile of rocks, but the rocks themselves are full of fissures (or cracks), and the macro porosity degrades over a period of ten years leaving a pile of finer material characteristic of the micro porous structure". It is usual to find relations of this theory with the law of Darcy, and the presentation of the theory involves displacement, pressure associated with the pores and pressure associated with the fissures [4, 7, 8].

Since the first works of Barenblatt et al. [1, 9], a large number of papers have been published dealing with mathematical issues as the existence and uniqueness of solutions or the energy decay (see, for instance, [2, 4, 7, 8, 10, 11, 12, 13, 14, 15]).

To describe the behaviour of porous solids materials some proposals have been stated. Nowadays, the theory proposed by Nunziato and Cowin [16] is commonly accepted as one of the non-classical elasticity theories. Grosso modo, it is supposed that in the materials there is a skeleton or material matrix that is elastic, and the interstices are voids in the material. A lot of contributions can be found dealing with this theory, even with applications to geological materials such as rocks and soils or to manufactured materials such as ceramics and pressed powders [17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27].

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Based on the theory of Nunziato and Cowin, and following a rational process (not just intuitive), Ieşan and Quintanilla [28] set a model where it is supposed the existence of two porous structures, one associated with the material pores and the other with the microporosity. The material skeleton supporting both structures and the interactions between them is described by the constitutive equations. Both structures have influences on the elastic deformations of the material matrix and also over the heat conduction through the material. That means the porous structures and the heat conduction are strongly coupled. This alternative approach is currently under research and several qualitative results have been obtained [17, 29, 30, 31]. Moreover, we can see our theory as a particular sub-case of the theory proposed in [32]. Notice that the theory we consider here coincides with the classical thermoelasticity theory if no porous structures are considered.

We want to highlight two issues. The first one is the novelty of the model, and the second one is that our approach is mainly theoretical. From our point of view, we believe that any theory needs a mathematical and physical analysis that allows to decide its applications to the real-world situations. Our paper is addressed in this line. We also want to remark the similarities, from a mathematical perspective, between the equations for elastic materials with double porosity and those for microstretch materials. That means that the equations that we study in this paper can also be viewed as the equations used to describe a mixture of microstretch materials if their macroscopic structures coincide.

Ieşan and Quintanilla [28] introduced only the thermal dissipation in their model. We consider dissipation also in the porous structures. To be precise, we will consider one dissipative mechanism on each porous structure.

We restrict our attention to the one-dimensional problem. This work is parallel to [33], where the existence of a unique solution and an energy decay property were proved. Here, we provide the numerical analysis of the corresponding variational problem, obtaining a discrete stability property, proving some a priori error estimates and performing some numerical simulations which show the behaviour of the solution.

The paper is outlined as follows. The mathematical model is briefly described in Section 2 following the parallel contribution [33], deriving its variational formulation. An existence and uniqueness result, and an energy decay property, are also stated. Then, in Section 3 a fully discrete approximation is introduced, based on the finite element method and the backward Euler scheme. A discrete stability property is proved, a priori error estimates are obtained for the approximative solutions and, under suitable regularity assumptions, the linear convergence of the algorithm is derived. Finally, some numerical simulations are presented in Section 4

2. The model and its variational formulation

In this section, following [33] we describe briefly the model, derive its variational formulation and state the main results (see [33] for further details).

Let us denote by $[0,\ell]$, $\ell > 0$, and $[0,T]$, $T > 0$, the one-dimensional rod of length ℓ and the time interval of interest, respectively. Moreover, let $x \in [0,\ell]$ and $t \in [0,T]$ be the spatial and time variables. In order to simplify the writing, we do not indicate the dependence of the functions on x and t, the time derivatives are denoted by one (first-order) or two (second-order) dots over a variable and the subscript x under a variable represents its spatial derivative.