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AN UNCONDITIONALLY STABLE NUMERICAL SCHEME FOR A COMPETITION SYSTEM INVOLVING DIFFUSION TERMS

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Abstract. A system of difference equations is proposed to approximate the solution of a system of partial differential equations that is used to model competing species with diffusion. The approximation method is a new semi-implicit finite difference scheme that is shown to mimic the dynamical properties of the true solution. In addition, it is proven that the scheme is uniquely solvable and unconditionally stable. The asymptotic behavior of the difference scheme is studied by constructing upper and lower solutions for the difference scheme. The convergence rate of the numerical solution to the true solution of the system is also given.

Key words. Competing species, convergence, asymptotic behavior, implicit finite difference scheme.

1. Introduction

We consider the following system of nonlinear parabolic partial differential equations used to model dynamic population distribution or biomass of two species that are competing for resources while each undergoes diffusion:

$$p_t = m_1 \triangle p + a_1 p - b_1 p^2 - c_1 pq \qquad (t > 0,\ \mathbf{x} \in \Omega), \tag{1}$$

$$q_t = m_2 \triangle q + a_2 q - b_2 q^2 - c_2 pq \qquad (t > 0,\ \mathbf{x} \in \Omega), \tag{2}$$

$$\frac{\partial p}{\partial \eta}|_{\partial\Omega} = 0,\ \frac{\partial q}{\partial \eta}|_{\partial\Omega} = 0 \qquad (t > 0), \tag{3}$$

$$p(0, \mathbf{x}) = p_0(\mathbf{x}),\ q(0, \mathbf{x}) = q_0(\mathbf{x}) \qquad (\mathbf{x} \in \Omega). \tag{4}$$

Here, $p(t, \mathbf{x})$ and $q(t, \mathbf{x})$ denote the time-dependent populations of the two species, $\Omega \in \mathbb{R}^n$ is a bounded domain with outward normal η along the boundary. The Neumann boundary conditions suggest absence of migration. There is a substantial body of work about this system, where many properties of the solutions are extracted, including such considerations as coexistence and long-term population behaviors of the competing species; see, for example, [1], [2], [3], and [6] and references therein. If $c_1 = c_2 = 0$, each equation in the paired system has the form of a so-called Fisher's equation. Ways to approach the numerical solutions of these equations can be found in [4] and [5].

For a numerical approximation of (1)-(4), the author in [8] proposes a discretization that gives rise to a fully implicit finite difference scheme. For $\Omega \subset \mathbb{R}$, this takes on the form

$$\frac{p_i^{k+1} - p_i^k}{\triangle t} = m_1 \left[\frac{p_{i+1}^{k+1} - 2p_i^{k+1} + p_{i-1}^{k+1}}{(\triangle x)^2} \right] + p_i^{k+1}(a_1 - b_1 p_i^{k+1} - c_1 q_i^{k+1}) \tag{5}$$

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$$\frac{q_i^{k+1}-q_i^k}{\triangle t}=m_2\left[\frac{q_{i+1}^{k+1}-2q_i^{k+1}+q_{i-1}^{k+1}}{(\triangle x)^2}\right]+q_i^{k+1}(a_2-b_2q_i^{k+1}-c_2p_i^{k+1}) \tag{6}$$

where $p_i^k=p(k\triangle t,i\triangle x)$, $q_i^k=q(k\triangle t,i\triangle x)$.

The author in [8] then used Picard iteration to construct sequences of decreasing upper solutions $\{\bar{p}_i^k\}^m$ and $\{\bar{q}_i^k\}^m$, and of increasing lower solutions $\{\underline{p}_i^k\}^m$ and $\{\underline{q}_i^k\}^m$, such that if the time mesh size $\triangle t$ is chosen sufficiently small,

$$\lim_{m\to\infty}\{\bar{p}_i^k\}^m=\lim_{m\to\infty}\{\underline{p}_i^k\}^m=p_i^k \quad\text{and}$$
$$\lim_{m\to\infty}\{\bar{q}_i^k\}^m=\lim_{m\to\infty}\{\underline{q}_i^k\}^m=q_i^k.$$

To study the asymptotic behavior of the numerical solution, the author in [8] studied the steady state solution of (5)-(6), or the solutions of the nonlinear algebraic system

$$m_1\left[\frac{p_{i+1}-2p_i+p_{i-1}}{(\triangle x)^2}\right]+a_1p_i-b_1(p_i)^2-c_1p_iq_i=0. \tag{7}$$

$$m_2\left[\frac{q_{i+1}-2q_i+q_{i-1}}{(\triangle x)^2}\right]+a_2q_i-b_2(q_i)^2-c_2p_iq_i=0. \tag{8}$$

In [9]-[12], the author studies the result under the conditions where the minimal solution $(\underline{p}_i^*,\underline{q}_i^*)$ is equal to the maximal solution $(\bar{p}_i^*,\bar{q}_i^*)$ of (7)-(8). If this is the case, the author shows that $(p_i^k,q_i^k)\to(\underline{p}_i^*,\underline{q}_i^*)=(\bar{p}_i^*,\bar{q}_i^*)$.

The fully implicit scheme proposed in [8] conserves the dynamic properties of the system (1)-(4). The author in [8] also applied this method in [13] for a coupled system of quasilinear elliptic equations. However, it takes a significant amount of time to approximate the numerical solution using Picard iteration. In addition, it is hard to estimate the convergence rate as $\triangle t$ and $\triangle \mathbf{x}$ approach zero. Finally, $\triangle t$ must be chosen sufficiently small to guarantee convergence of the fully implicit system to the theoretical solution.

In this paper, we develop a new method for numerical approximation of the true solution (p,q) to (1)-(4); call this numerical approximation (p^k,q^k) for the time being. We propose a nonstandard finite difference method for discretizing the system that ends up requiring that a semi-implicit system of difference equations be solved for (p^k,q^k) rather than a fully implicit system as in [8]. We find the numerical solution to the system of difference equations directly. Then, fully independent of the choice of $\triangle t$, we prove the nonnegativity of p^k and q^k, the stability of the difference scheme, and that (p^k,q^k) converges to the true solution (p,q) of the system. We also show its rate of convergence to be $\mathcal{O}(\Delta t+\Delta x^2)$. We construct an upper solution $(\bar{p}^k,\bar{q}^k)$ and a lower solution $(\underline{p}^k,\underline{q}^k)$ to the system of difference equations using a related system of ordinary differential equations. Having constructed these upper and lower solutions, we will then be able to give a sufficient condition for coexistence of solutions of the system of difference equations and to provide a complete analysis of the long-term behavior of the numerical solution to (1)-(4).

In Section 2, we will introduce the difference scheme used for the approximation of (1)-(4) for $\Omega\subset\mathbb{R}$. We prove existence of the numerical solution to the scheme, and that it is stable independent of the choice of Δt and $\Delta \mathbf{x}$. We finish by giving the convergence rate of the numerical scheme to the true solution. In Section 3, we give more properties of the asymptotic behavior of the numerical solutions to