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PROVABLY SIZE-GUARANTEED MESH GENERATION WITH SUPERCONVERGENCE

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Abstract. The mesh conditions of high-quality grids generated by bubble placement method (BPM) and their superconvergence properties are studied in this paper. A mesh condition that for each pair of adjacent triangles, the lengths of any two oppsite edges differ only by a high order of the parameter h is derived. Furthermore, superconvergence estimations are analyzed on both linear and quadratic finite elements for elliptic boundary value problems under the above mesh condition. In particular, the mesh condition is found to be applicable to many known superconvergence estimations under different types of equations. Finally, numerical examples are presented to demonstate the superconvergence properties on BPM-based grids.

Key words. Bubble placement method, mesh condition, superconvergence estimation.

1. Introduction

Superconvergence of finite element solutions to partial differential equations has been studied intensively for many decades [1, 2, 3, 6]. It is shown to be an important tool to develop high-performance finite elements. The superconvergence property can significantly improve the accuracy of finite element solution and its derivatives with few extra calculation and storage. And it is mainly used to construct a posteriori error indicator [3].

The existing research work basically follows two approaches. One is to find the super-close point of finite element interpolation approximation, and then use the interpolation weak estimation to obtain the superconvergence properties of finite element solution and its derivatives [4, 5]. Another is to obtain superconvergence properties by various post-processing techniques, including weighted averaging, local L^2-projection, extrapolation, and gradient recovery methods. In particular, gradient recovery methods have achieved great success in numerical simulations in engineering problems, such as the popular superconvergent patch recovery (SPR) method [6, 7, 8] and the polynomial preserving gradient recovery (PPR) method [24].

However, in early superconvergence theory, specially structured grids were normally required, such as the strongly regular grids composed of equilateral triangles [9], which brought a great difficulty to mesh generation techniques. Thus a consensus was hardly reached between theory of superconvergence and mesh generation.

Recently, several studies have striven to relieve this issue. From one hand, superconvergence theory was well developed, though under assumed mesh conditions. In particular, Bank and Xu [10, 11] studied superconvergence on mildly structured grids where most pairs of elements form an 'approximate parallelogram'. They also

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proved that linear finite element solution is superclose to its linear interpolant of exact solution. From this work, Xu and Zhang [12] established the superconvergence estimations for several post-processing techniques. Further, Huang and Xu investigated superconvergence properties of quadratic triangular element on mildly structured grids [9]. From the other hand, superconvergence phenomena have existed in several mesh generation algorithms. For example, the centroidal Voronoi tessellation (CVT)-based methods have been successfully applied to develop high-quality grids [13]. However, its superconvergence estimations from some certain mesh conditions were not clearly provided [14].

In recent years, the so called bubble placement method (BPM) has been systematically studied by Nie et. al. [15, 16, 17]. The advantage of BPM is to generate high-quality grids on many complexly bounded 2D and 3D domains, and BPM can be easily used in adaptive finite element method and anisotropic problems [18, 19, 20, 21, 22, 30]. In addition, due to the natural parallelism of BPM, computational efficiency has been improved greatly to solve large-scale problems [23]. Yet, superconvergence on BPM-based grids has not been fully explored. The goal of this paper is to analyze a mesh condition on BPM-based grids, such that superconvergence results can be obtained both theoretically and numerically.

In this paper, we will carefully investigate the superconvergence properties on BPM-based grids. Our work is mainly composed of two parts: in the first part, a mesh condition associated with element edge length and desired length is derived for BPM-based grids; the second part presents two superconvergence results for linear and quadratic finite elements, respectively. These superconvergence results can be used to construct posteriori error estimates under gradient recovery operators.

The rest of this paper is organized as follows. Section 2 gives the derivation of mesh conditions for BPM-based grids. Superconvergence estimations on linear and quadratic finite elements are analyzed in Section 3. Numerical experiments on elliptic boundary value problem with some typical computational domains are given in Section 4 and further discussed in Section 5. Conclusions and future works are summarized in Section 6.

2. Mesh conditions

2.1. BPM. Bubble placement method was originally inspired by the idea of bubble meshing [25, 26] and the principle of molecular dynamics. The computational domain is regarded as a force field with viscosity, and bubbles are distributed in this domain. Each bubble is driven by interaction forces from its adjacent bubbles, expressed as [27]:

$$f(w) = \begin{cases} k_0 \left(1.25w^3 - 2.375w^2 + 1.125\right) & 0 \le w \le 1.5 \\ 0 & 1.5 < w. \end{cases} \tag{1}$$

The output of bubble centers are denoted as nodes in the computational domain, where $w = \frac{l_{ij}}{\bar{l_{ij}}}$, l_{ij} is the actual distance between bubble i and bubble j, $\bar{l_{ij}}$ is the user-defined distance. The motion of each bubble satisfies the Newton's second law of motion. BPM can be mainly divided into 3 steps: initialization, dynamic simulation, bubble insertion and deletion operations. And BPM is regarded to be controlled by two nested loops, which is schematically illustrated in Fig. 1.