INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 17, Number 3, Pages 297–315

ANALYSIS OF A GALERKIN FINITE ELEMENT METHOD APPLIED TO A SINGULARLY PERTURBED REACTION-DIFFUSION PROBLEM IN THREE DIMENSIONS

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Abstract. We consider a linear singularly perturbed reaction-diffusion problem in three dimensions and its numerical solution by a Galerkin finite element method with trilinear elements. The problem is discretised on a Shishkin mesh with N intervals in each coordinate direction. Derivation of an error estimate for such a method is usually based on the (Shishkin) decomposition of the solution into distinct layer components. Our contribution is to provide a careful and detailed analysis of the trilinear interpolants of these components. From this analysis it is shown that, in the usual energy norm the errors converge at a rate of $\mathcal{O}(N^{-2} + \varepsilon^{1/2} N^{-1} \ln N)$. This is validated by numerical results.

Key words. Reaction-diffusion, finite element, Shishkin mesh, three-dimensional.

1. Introduction

Consider the following three-dimensional singularly perturbed reaction-diffusion problem posed on the unit cube

$$
\begin{aligned}
Lu &:= -\varepsilon^2 \Delta u + bu = f \quad &&\text{in } \Omega := (0,1)^3, \\
u &= 0 \quad &&\text{on } \partial\Omega,
\end{aligned} \tag{1}
$$

where the reaction coefficient $b(x, y, z) \geq 2\beta^2$, and β is a positive constant. In the case of interest, the diffusion parameter, ε, can be arbitrarily small, i.e. $0 < \varepsilon \ll 1$, and so the problem (1) is singularly perturbed. As such, its solution typically exhibits layers of width $\mathcal{O}(\varepsilon)$ along the boundary of the domain, $\partial\Omega$.

It is well known that computing numerical solutions to singularly perturbed problems presents many difficulties. Solutions to these problems tend to be anisotropic in nature on regions along the boundary or interior of the domain. Over the course of the past five decades there have been many advances in devising specialised numerical schemes to deal with this phenomenon. Many of these schemes fall into the category of "*fitted mesh*" methods, where a standard discretisation is applied on a specially designed non-uniform mesh.

The idea of using non-uniform meshes to solve singularly perturbed problems was first introduced in 1969 by Bakhvalov [4]. However, it wasn't until the early 1990s, with the introduction of the piecewise uniform mesh of Shishkin [19, 20], that fitted mesh methods gained major prominence in the literature. In particular, the application of finite difference methods to the one- and two-dimensional analogues of (1) is well understood, see, e.g., [7, 17].

Finite element methods (FEMs) for one- and two-dimensional singularly perturbed reaction-diffusion problems, discretised on Shishkin meshes, are also well documented in the literature. In the one-dimensional setting, Sun and Stynes [22]

Received by the editors July 26, 2017 and, in revised form, January 6, 2020.
2000 *Mathematics Subject Classification.* 65N12, 65N15, 65N30.

prove almost optimal convergence using piecewise polynomial elements, see also [10, Thm. 6.6]. In two dimensions, Li and Navon [8] provide the first numerical analysis for a Galerkin FEM applied to a singularly perturbed reaction-diffusion problem. They prove almost second-order convergence in the L^2-norm. Using piecewise polynomial elements of order k, and quantifying the error in the energy norm, Apel [3] proves convergence of $\mathcal{O}(N^{-k-1} + \varepsilon^{1/2}N^{-k}(\ln N)^{k+1})$. Using piecewise bilinear elements, Liu et al. [11] prove convergence in the energy norm of $\mathcal{O}(N^{-2} + \varepsilon^{1/2}N^{-1}\ln N)$. In that same paper, the authors provide the first full numerical analysis of a sparse grid FEM applied to a singularly perturbed problem. For a two-scale sparse grid method applied to a reaction-diffusion problem they prove that the method converges at the same rate as the standard Galerkin FEM. Madden and Russell [15]. extend the results of [11] and prove convergence of a multiscale sparse grid FEM applied to the same problem.

The analysis of three-dimensional singularly perturbed reaction-diffusion problems has received, comparatively, little attention. The work of most relevance to this paper is that of Shishkin and Shishkina [21], which provides a valuable solution decomposition in three dimensions. Crucial bounds on the derivatives of each of the solution decomposition components are also given. Chadha and Kopteva [5] provide maximum norm *a posteriori* error estimates for a finite difference method applied to a semi-linear reaction-diffusion problem in three dimensions. We also make note of the work of Lopez and co-authors [12, 13, 14].

The numerical solution of three dimensional problems is computationally demanding, which motivates the use of sparse grid techniques, such as those referred to above. In particular, the authors have devised and analysed a two-scale sparse grid method for (1) [18]. However, that analysis establishes the difference between the sparse grid and standard Galerkin solutions, so, for completeness, a full analysis of the latter method is required; this paper provides that. It does so by employing a Shishkin decomposition of the solution into distinct layer components, established by Shishkin and Shishkina [21]. We then provide a careful and detailed analysis of the trilinear interpolants of these components. From this, the numerical analysis of the method follows. It should be noted that the interpolation results given here also form the basis for the analysis of other techniques, such as the balanced norm method of Lin and Stynes [9] and the FOSLS-type method of Adler et al. [1].

This paper is organised as follows. In Section 2.1 we present a canonical example of a solution to (1), which motivates a three-dimensional piecewise uniform mesh. The Shishkin decomposition is presented in Section 2.2, and standard bounds on the components are reported. The heart of this paper is in Section 3, where we give a detailed analysis of the decomposition components in the energy norm. The FEM is given in Section 4, and the error estimate is readily deduced from the preceding interpolation analysis. Numerical results, presented in Section 5 verify that the theoretical results are sharp.

Notation. We use the following standard notation for function spaces and norms (see, e.g., [6]):

- let $\mathcal{C}^p(\Omega)$ be the space of all real-valued functions, v, defined on Ω, such that v, and all its partial derivatives up to order p, are continuous on Ω;
- $\mathcal{C}^{m,\alpha}(\Omega) = \{v \in \mathcal{C}^m(\bar{\Omega}); \forall\beta, |\beta| = m, \exists C_\beta, \forall x, y \in \Omega, |\partial^\beta v(x) - \partial^\beta v(y)| \le C_\beta \|x - y\|^\alpha\}$;