INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 17, Number 3, Pages 332–349

ANALYSIS OF ARTIFICIAL DISSIPATION OF EXPLICIT AND IMPLICIT TIME-INTEGRATION METHODS

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Abstract. Stability is an important aspect of numerical methods for hyperbolic conservation laws and has received much interest. However, continuity in time is often assumed and only semidiscrete stability is studied. Thus, it is interesting to investigate the influence of explicit and implicit time integration methods on the stability of numerical schemes. If an explicit time integration method is applied, spacially stable numerical schemes for hyperbolic conservation laws can result in unstable fully discrete schemes. Focusing on the explicit Euler method (and convex combinations thereof), undesired terms in the energy balance trigger this phenomenon and introduce an erroneous growth of the energy over time. In this work, we study the influence of artificial dissipation and modal filtering in the context of discontinuous spectral element methods to remedy these issues. In particular, lower bounds on the strength of both artificial dissipation and modal filtering operators are given and an adaptive procedure to conserve the (discrete) $\mathbf{L}_2$ norm of the numerical solution in time is derived. This might be beneficial in regions where the solution is smooth and for long time simulations. Moreover, this approach is used to study the connections between explicit and implicit time integration methods and the associated energy production. By adjusting the adaptive procedure, we demonstrate that filtering in explicit time integration methods is able to mimic the dissipative behavior inherent in implicit time integration methods. This contribution leads to a better understanding of existing algorithms and numerical techniques, in particular the application of artificial dissipation as well as modal filtering in the context of numerical methods for hyperbolic conservation laws together with the selection of explicit or implicit time integration methods.

Key words. Hyperbolic conservation laws, flux reconstruction, summation-by-parts, artificial viscosity, full discrete stability, time integration methods.

1. Introduction

Stability is one of the main desirable properties for a numerical scheme to solve hyperbolic conservation laws. This is due to the fact that at least for linear symmetric systems, an energy estimate (and the correct number of boundary conditions for initial boundary value problems) comes along with uniqueness and existence of a solution [21]. In the last years, several approaches have been proposed to construct entropy stable/conservative schemes like in [2, 3, 8, 9, 11, 13, 36, 40, 46, 52, 54, 56] and references therein. Recently, Abgrall [2] presented a way to build entropy stable/conservative schemes using the Residual Distribution (RD) framework. In [4], this idea is extended to Flux Reconstruction (FR) methods. This idea is fairly general and has been extended and re-interpreted in the discontinuous Galerkin (DG) context in [5]. However, besides the spacial discretization, the selection of the time integration method is essential for stability of these methods.

First of all, one has to choose between explicit or implicit methods to march forward in time. Implicit methods have favorable stability properties and, in particular, allow larger time steps. For instance, by using implicit time integration methods build on Summation-By-Parts (SBP) operators in time[1] [34], the semidiscrete stability results transfer directly to the fully discrete case [12, 32, 33]. It should be stressed, however, that

Received by the editors December 16, 2018 and, in revised form, December 15, 2019.
2000 *Mathematics Subject Classification.* 65M12, 65M20.

[1]These schemes can be interpreted as implicit Runge-Kutta (RK) methods [6, 43].

implicit methods yield to (typically non-)linear systems to be solved. Since the time step is also constrained by accuracy requirements, implicit methods may not be as efficient as explicit ones.

Explicit time integration methods, on the other hand, can be faster and easier to implement, but suffer from stability issues triggered by additional error terms. One way to improve the stability properties of numerical schemes is the usage of artificial dissipation. This idea dates back to early works of von Neumann and Richtymer [55]. Since then, many researchers have contributed to the development and extension of artificial dissipation methods, including the works [29, 31, 44, 52, 53].

In this work, we investigate the connections between artificial dissipation in explicit time integration methods and implicit time integration methods without additional limiting from point of stability. We further extend this investigation to modal filtering. Modal filtering is strongly connected to artificial dissipation methods in spectral and spectral element methods [7, 17, 18, 25, 30, 38] and provides an alternative which, in some cases[2], might be more efficient and easier to implement. In particular, we demonstrate that it is possible to mimic the dissipation (and thus stability) inherent in implicit time integration methods for explicit time integration methods when modal filtering with a suitable filter strength is incorporated. This result directly carries over to explicit time integration methods with suitable artificial dissipation terms. Thus, we are able to present an approach to obtain stable fully discrete schemes using explicit time integration. Such discretizations combine the favorable stability properties of implicit time integration methods with the efficiency gain of explicit time integration methods. Finally, we would like to mention that recently a relaxation Runge-Kutta approach has been proposed to construct fully discrete explicit energy (entropy) conservative/stable schemes in [23, 48]. Their approach has some similarities to our consideration but instead of working with modal filters or artificial viscosity to destroy the energy production in time, they change the final update step in the RK method to guarantee that the discrete energy equality is fulfilled.

For sake of simplicity, the explicit Euler method is considered. Yet, at least for nonlinear problems, the same stability issues arise for strong stability preserving (SSP) RK schemes, since they can be written as convex combination of explicit Euler steps [19]. In the appendix, we show how our investigation carries over to general Runge-Kutta methods. Recent relevant articles concerned with the strong stability of explicit Runge-Kutta methods are, e.g., [27, 28, 42, 45, 50, 51].

The rest of this work is organized as follows: In section 2, we start by briefly revisiting the FR method in its formulation using SBP operators. This method yields a stable semidiscretisation and thus serves as a representative of a stable scheme. Yet, the examinations are rather general and valid for other spacial discretizations as well. In section 3, we investigate the mechanism which triggers stability issues when semidiscretisation (even stable ones) are evolved in time by explicit time marching. Further, we investigate the stabilizing effect of artificial dissipation terms and modal filtering. In principle, similar investigations are well-known. Performing this analysis in a vector matrix-vector representation including suitable discrete inner products, however, we are able derive new (strict) bounds on the artificial viscosity strength and filter strength for stability to carry over in time. Building up on this strategy, adaptive filtering strategies can be derived which yield methods with neither not enough nor too much dissipation. This might be beneficial in smooth regions for long time simulations. Section 4 explores the connection between implicit time integration and modal filtering in explicit time integration. We

[2]For instance, when the method is already formulated in a suitable modal basis.