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EFFICIENT GALERKIN-MIXED FEMS FOR INCOMPRESSIBLE MISCIBLE FLOW IN POROUS MEDIA

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Abstract. The paper focuses on numerical study of the incompressible miscible flow in porous media. The proposed algorithm is based on a fully decoupled and linearized scheme in the temporal direction, classical Galerkin-mixed approximations in the FE space $(V_h^r, S_h^{r-1} \times \mathbf{H}_h^{r-1})$ $(r \geq 1)$ in the spatial direction and a post-processing technique for the velocity/pressure, where V_h^r and $S_h^{r-1} \times \mathbf{H}_h^{r-1}$ denotes the standard C^0 Lagrange FE and the Raviart-Thomas FE spaces, respectively. The decoupled and linearized Galerkin-mixed FEM enjoys many advantages over existing methods. At each time step, the method only requires solving two linear systems for the concentration and velocity/pressure. Analysis in our recent work [37] shows that the classical Galerkin-mixed method provides the optimal accuracy $O(h^{r+1})$ for the numerical concentration in L^2-norm, instead of $O(h^r)$ as shown in previous analysis. A new numerical velocity/pressure of the same order accuracy as the concentration can be obtained by the post-processing in the proposed algorithm. Extensive numerical experiments in both two- and three-dimensional spaces, including smooth and non-smooth problems, are presented to illustrate the accuracy and stability of the algorithm. Our numerical results show that the one-order lower approximation to the velocity/pressure does not influence the accuracy of the numerical concentration, which is more important in applications.

Key words. Galerkin-mixed FEM, incompressible miscible flow in porous media, fully linearized scheme.

1. Introduction

Numerical study for incompressible miscible flow in porous media plays an important role in many applications, such as reservoir simulations and surface contaminant transport and remediation. In these areas, the incompressible flow is described by the following miscible displacement system

$$\Phi \frac{\partial c}{\partial t} - \nabla \cdot (D(\mathbf{u}) \nabla c) + \mathbf{u} \cdot \nabla c = \hat{c} q^I - c q^P, \tag{1}$$

$$-\nabla \cdot \frac{\mathbf{K}(x)}{\mu(c)} \nabla p = q^I - q^P, \tag{2}$$

with the initial and boundary conditions:

$$\begin{aligned} &\mathbf{u} \cdot \mathbf{n} = 0,\ D(\mathbf{u}) \nabla c \cdot \mathbf{n} = 0 \quad \text{for } x \in \partial\Omega,\ t \in [0, T], \\ &c(x, 0) = c_0(x) \qquad\qquad\qquad\ \text{for } x \in \Omega, \end{aligned} \tag{3}$$

where $\mathbf{u}$ denotes the Darcy velocity of the fluid mixture defined by

$$\mathbf{u} = -\frac{\mathbf{K}(x)}{\mu(c)} \nabla p, \tag{4}$$

p is the pressure of the fluid mixture and c is the concentration. Moreover, here $\mathbf{K}(x)$ is the permeability tensor of the medium, $\mu(c)$ is the concentration-dependent viscosity, Φ is the porosity of the medium, q^I and q^P are the given injection and production sources, $\hat{c}$ is the concentration in the injection source, and $D(\mathbf{u}) =$

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$[D_{ij}(\mathbf{u})]_{d\times d}$ is the velocity-dependent diffusion-dispersion tensor, which may be given in different forms (see [4, 34] for details). We assume that the system is defined in a bounded domain Ω in $\mathbb{R}^d$ $(d = 2, 3)$ and $t \in [0, T]$.

In the last several decades, numerous effort has been devoted to the development of numerical methods for the system (1)-(4), *e.g.* see [18, 26, 28, 25] for Galerkin FEMs, [7, 9, 10, 13, 30] for Galerkin-mixed methods, [11, 15, 19, 20, 41] for characteristics type methods, [38, 39] for ELLAM and [1, 23, 36] for others. Numerical simulations have been made extensively in various engineering areas [10, 14, 17]. Two review articles for numerical methods in these areas were presented by Ewing and Wang [21] and Scovazzi et al.[35], respectively. Theoretical analysis for the existence of weak solutions of the system was studied by Feng [22], while the existence of classical solutions is unknown so far.

As the system consists of a parabolic concentration equation and an elliptic pressure equation, one should choose a proper approximation to each of these computational components, more precisely the degree of piecewise polynomial used for each component. Since the concentration is the most important physical component, the accuracy of numerical concentration is a major concern in applications. Existing algorithms are mainly based on previous error estimates which however may not be optimal in some sense. Therefore, these algorithms often requires a high computational cost and complicated implementation. Numerical analysis for the system (1)-(4) in two-dimensional space was first presented by Ewing and Wheeler [18] for a standard Galerkin-Galerkin approximation $(c_h, p_h) \in (V_h^r, \widehat{V}_h^k)$ where V_h^r denotes C^0 Lagrange finite element space of piecewise polynomials of degree r and $\widehat{V}_h^k := V_h^k/\{constant\}$. Further analysis for Galerkin-Galerkin FEMs can be found in literature [25, 32, 43]. Due to the nature of discontinuity of the gradient of the pressure and continuity of the Darcy velocity in applications, the Galerkin-mixed method is more popular in many areas. In this method, a standard C^0 Lagrange type approximation $c_h \in V_h^r$ is applied for the concentration equation and a mixed approximation in the Raviart–Thomas finite element space (or other mixed FE space) $(p_h, \mathbf{u}_h) \in S_h^k \times \mathbf{H}_h^k$ is used for the pressure equation. The error estimate was first presented in [12] for a semi-discrete Galerkin-mixed method and later, in [13] for a fully discrete semi-implicit Euler scheme. In [13], the error estimate

$$\|c^n - c_h^n\|_{L^2} + \|p^n - p_h^n\|_{L^2} + \|\mathbf{u}^n - \mathbf{u}_h^n\|_{L^2} \le C(\tau + h_c^{r+1} + h_p^{k+1}) \tag{5}$$

was established for $d = 2$ under the time step restriction $\tau = o(h)$ and an extra spatial mesh condition

$$h_c^{-1} h_p^{k+1} = o(1) \tag{6}$$

where h_c and h_p denote the mesh sizes of FE discretization for the concentration and pressure equations, respectively. Further studies on time step restriction and spatial mesh condition were presented in [7, 10, 26, 28, 32]. Analysis for many other methods can be found in literature [1, 6, 15, 38, 41]. Based on the error estimate (5), it was suggested in [13] to use the finite element space $V_h^r \times S_h^r \times \mathbf{H}^r$ $(k = r > 0)$ and later, such a combination of finite element spaces was used widely in computations. On the other hand, due to the discontinuity of physical parameters in applications, the most popular Galerkin-mixed method is the lowest order one $(r = 1, k = 0)$ [7, 9, 13, 15, 21, 35], *i.e.*, a linear approximation to the concentration and the zero-order Raviart–Thomas approximation to the pressure and velocity. The lowest order Galerkin-mixed method has been widely used in a variety of numerical simulations, *e.g.*, see [13, 17, 19, 35]. In this case, the error estimate (5)