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A FINITE DIFFERENCE METHOD FOR STOCHASTIC NONLINEAR SECOND-ORDER BOUNDARY-VALUE PROBLEMS DRIVEN BY ADDITIVE NOISES

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Abstract. In this paper, we present a finite difference method for stochastic nonlinear second-order boundary-value problems (BVPs) driven by additive noises. We first approximate the white noise process with its piecewise constant approximation to obtain an approximate stochastic BVP. The solution to the new BVP is shown to converge to the solution of the original BVP at $\mathcal{O}(h)$ in the mean-square sense. The approximate BVP is shown to have certain regularity properties which are not true in general for the solution to the original stochastic BVP. The standard finite difference method for deterministic BVPs is then applied to approximate the solution of the new stochastic BVP. Convergence analysis is presented for the numerical solution based on the standard finite difference method. We prove that the finite difference solution converges to the solution to the original stochastic BVP at $\mathcal{O}(h)$ in the mean-square sense. Finally, we perform several numerical examples to validate the theoretical results.

Key words. Stochastic nonlinear boundary-value problems, finite difference method, additive white noise, mean-square convergence, order of convergence.

1. Introduction

In this paper, we investigate the convergence properties of a finite difference method applied to scalar stochastic nonlinear second-order boundary-value problems (BVPs) driven by additive white noises. More specifically, we are interested in the stochastic BVP (SBVP)

$$u'' = f(x,u) + g(x)\dot{W}(x), \quad x \in (a,b), \quad u(a) = \alpha, \quad u(b) = \beta, \tag{1}$$

where $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are given functions. Here, α and β are deterministic real constants and $\dot{W}$ is the white noise. The white noise is a generalized function or a distribution and it can be written informally as $\dot{W}(x) = \frac{dW(x)}{dx}$ in the sense of distribution. Here, $W(x)$ is the one-dimensional standard Brownian motion (or Wiener process) which is defined on a complete probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\{\mathcal{F}_x\}_{a \le x \le b}$ satisfying the usual conditions (*i.e.*, the filtration is right-continuous and contains all P-null sets in $\mathcal{F}$) and carrying a standard one-dimensional Brownian motion W. We note that the stochastic process $W = W(x),\ x \in [a,b]$ has the following important properties:

(1) $W(a) = 0$ with probability one.
(2) The trajectories (or sample paths) $x \to W(x)$ are continuous for $x \in [a,b]$.
(3) For every $a \le x < y \le b$, the increment $W(y) - W(x)$ is normally distributed with mean 0 and variance $y - x$. Symbolically, we write $W(y) - W(x) \sim \mathcal{N}(0, y-x)$.
(4) $W(x)$ has independent increments *i.e.*, for every partition $a = x_0 \le x_1 < \cdots < x_N = b$, the increments $\Delta W_i = W(x_i) - W(x_{i-1}),\ i = 1, 2, \ldots, N$, are independent.

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Although Brownian paths are not differentiable pointwise, we may interpret their derivative in a distributional sense to get a generalized stochastic process called white noise $\dot{W} = \frac{dW}{dx}$. The term "white noise" arises from the spectral theory of stationary random processes, according to which white noise has a "flat" power spectrum that is uniformly distributed over all frequencies (like white light). This can be observed from the Fourier representation of Brownian motion.

In our analysis, we assume that the SBVP (1) has a unique solution. The existence and uniqueness of the solution to SBVPs were established by Nualart and Pardoux in [29, 30]. We further assume that the function g is continuous on $[a, b]$ and satisfies the uniform Lipschitz condition with Lipschitz constant L_g:

$$|g(x) - g(y)| \leq L_g |x - y|. \tag{2}$$

Finally, we assume that the nonlinear function $f(x, u)$ satisfies the following conditions

(1) $f(x, u)$ and $f_u(x, u)$ are continuous functions on the set $D = \{(x, u) \mid x \in [a, b],\ u \in \mathbb{R}\}$,
(2) there exist constants K_1 and K_2 such that

$$0 < K_1 \leq f_u(x, u) \leq K_2, \quad \text{for all } (x, u) \in D. \tag{3}$$

Using the Mean-Value Theorem, it follows that f satisfies the following uniform Lipschitz condition on D in the variable u with uniform Lipschitz constant $L_f = K_2$

$$|f(x, u) - f(x, v)| \leq L_f |u - v|, \quad \text{for all } (x, u),\ (x, v) \in D = [a, b] \times \mathbb{R}. \tag{4}$$

We remark that (1) is a formal notation due to poor regularity of the white noise. A solution to the SBVP (1) is defined in terms of integral equations. To define the solution u, we first introduce a new variable $v = u'$. Then we convert (1) into the system

$$u' = v, \quad v' = f(x, u) + g(x)\dot{W}(x), \quad x \in (a, b), \quad u(a) = \alpha, \quad u(b) = \beta. \tag{5}$$

The stochastic process $(u, v) \in \mathbb{R}^2$ is a solution to (5) if (u, v) satisfies the integral equations

$$u(x) = u(a) + \int_a^x v(y)dy, \quad x \in (a, b), \tag{6a}$$

$$v(x) = v(a) + \int_a^x f(y, u(y))\, dy + \int_a^x g(y)dW(y), \quad x \in (a, b), \tag{6b}$$

with the boundary conditions $u(a) = \alpha$ and $u(b) = \beta$. The integral in (6a) and the first integral in (6b) are pathwise Riemann integrals. However, the last integral in (6b) is an Itô stochastic integral. Since the Brownian paths are of unbounded variation on $[a, x]$ for every $x > a$, the latter integral cannot be defined as a Riemann-Stieltjes integral.

Stochastic differential equations (SDEs) are used to describe more realistic models. They provide suitable mathematical tools to model real-world problems with uncertainties that may be originated from various sources such as side (initial and boundary) conditions, geometry representation of the domain, and input parameters. Many areas of applications use SDEs including physics, biology, finance, economics, insurance, signal processing and filtering, population dynamics, and genetics; see for examples [17, 25, 31, 32, 33, 34, 38].

Unlike deterministic BVPs, there are very few SDEs with exact analytical solutions. Therefore, numerical methods are usually necessary to approximate their solutions.