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SOME NEW DEVELOPMENTS OF POLYNOMIAL PRESERVING RECOVERY ON HEXAGON AND CHEVRON PATCHES

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Abstract. Polynomial Preserving Recovery (PPR) is a popular post-processing technique for finite element methods. In this article, we propose and analyze an effective linear element PPR on the equilateral triangular mesh. With the help of the discrete Green's function, we prove that, when using PPR to the linear element on a specially designed hexagon patch, the recovered gradient can reach $O(h^4|\ln h|^{\frac{1}{2}})$ superconvergence rate for the two dimensional Poisson equation. In addition, we apply PPR to the quadratic element on uniform triangulation of the Chevron pattern with an application to the wave equation, which further verifies the superconvergence theory.

Key words. Finite element method, post-processing, gradient recovery, superconvergence.

1. Introduction

In recent years, since the development of the high accuracy post-processing and a posteriori error estimate ([1] and [2]), there has been growing interest in the superconvergence and other kinds of high accuracy methods such as defect correction and extrapolation. Finite element recovery techniques are post-processing methods that reconstruct numerical approximations from finite element solutions to achieve better results. We consider only C^0 finite element methods, although generalization to other finite element methods, such as non-conforming and discontinuous Galerkin methods, are feasible. Let u be a solution of certain differential equation, and u_h be the finite element approximation of u. The goal of a recovery technique is to construct $G_h u_h$ based on u_h such that $G_h u_h$ is a better approximation of ∇u than ∇u_h. Naturally, the mathematical background of recovery techniques is closely related to the finite element superconvergence theory, see, e.g., the monographs [3] and [4].

Zienkiewicz and Zhu first introduced the gradient recovery method Superconvergence Patch Recovery (SPR, ([5]) in 1992 based on a local discrete least-squares fitting. Later, Zhang and Naga proposed an alternative strategy ([6]) called Polynomial Preserving Recovery (PPR) to recover the gradient. Theoretical analysis reveals that PPR has better superconvergence properties than SPR ([7]). It has been implemented by commercial finite element software COMSOL Multiphysics as a superconvergence tool. There have been further developments on applications of PPR in numerical methods. For example, Guo and Yang ([8]) generalized the study of PPR to high-frequency wave propagation in 2016. Wang etc.al establish the superconvergence for Maxwell equations and combine with PPR that leads to global superconvergence for recovered quantities in energy norms ([9]). Du and Zhang study the superclosesness property of the linear Discontinous Galerkin finite element method and its superconvergence behavior after post-processing by the PPR ([10]). Guo et al.generalized the idea of PPR to the general polygons, which only uses the degrees of freedom and has the consistency on arbitrary polygonal meshes

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by the polynomial preserving property([11]). They prove the polynomial preserving and boundedness properties of the generalized gradient recovery operator.

In practice, PPR is performed on an element patch ω_z (around z) which is a union of elements that covers all nodes needed for the construction of $G_h u_h(z)$. Different mesh patterns and selection of patches result in different recovery. Some popular mesh patterns include the regular pattern, the Chevron pattern, the Union-Jack pattern, etc. ([6] and [7]). In general, PPR can attain h^2 superconvergence rate for the recovered gradient at an element vertex z for the linear element (Theorem 8.17 of [7]). In this article, we design a hexagon patch on equilateral triangulation (Section 3.1) to reach a surprising superconvergence rate $h^4|\ln h|^{\frac{1}{2}}$ for the recovered gradient from the linear element (Theorem 7). Standard approximation theory fails to prove such a higher order superconvergence. In order to prove our theory, we use the asymptotic error expansion in [12] and interior maximum norm estimates for the discrete Green's function in Section 3.2. Furthermore, an equal superconvergence phenomenon is found on equilateral triangulation (Theorem 8). In addition, we apply PPR to the quadratic element on the uniform triangulation of the Chevron pattern, which further verifies the superconvergence stated in Theorem 3.1 in [6]. We also perform the quadratic PPR numerical experiments for a wave equation on the Chevron pattern mesh.

An outline of this paper is as follows. We devote Section 2 to existed theory for PPR. The general set up for the linear element PPR on the Hexagon patch is then constructed in Section 3. Finally the applications of the PPR to the quadratic element on the uniform triangulation of the Chevron pattern are presented in Section 4.

2. Some preliminaries of PPR

In this section, we introduce some basic knowledge of PPR in 2D. We consider the following variational problem on a polygonal domain Ω : Find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = f(v), \quad \forall v \in H_0^1(\Omega), \tag{1}$$

where

$$a(u, v) = \int_\Omega [(A\nabla u + bu) \cdot \nabla v + cuv].$$

We assume that all the coefficient functions are smooth, A is a 2×2 symmetric positive definite matrix, $f(.)$ is a linear functional, and the bilinear form is continuous and satisfies the inf-sup condition (8.3.14)-(8.3.15) of [7] on $H^1(\Omega)$.

Let $T_h = \{K\}$ be a finite regular triangulation of Ω of width h with all its boundary vertices on $\partial\Omega$. Corresponding to T_h, we define the following finite element spaces:

$S_h(\Omega) = \{v_h \in C(\Omega_h) : v_h$ is piecewise polynomial of degree$\leq k$ on each $K \in T_h\}$
$S_h^0(\Omega) = \{v \in S_h : supp(v) \in \Omega_h\}$.

where $\Omega_h = \cup\{K \in T_h\}$. Then the finite element approximation $u_h \in S_h^0(\Omega)$ satisfies

$$a(u_h, v) = f(v), \quad \forall v \in S_h^0(\Omega_h). \tag{2}$$

To ensure the uniqueness of the finite element solution, we assume the discrete inf-sup condition (8.3.17) of [7].

Given a node z, we select $n \geq (k+2)(k+3)/2$ sampling points adjacent to z, and fit a polynomial of degree $k+1$, in the least square sense, with values of u_h at