INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 17, Number 4, Pages 532–542

ERROR ESTIMATES IN BALANCED NORMS OF FINITE ELEMENT METHODS FOR HIGHER ORDER REACTION-DIFFUSION PROBLEMS

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Abstract. Error estimates of finite element methods for reaction-diffusion problems are often realised in the related energy norm. In the singularly perturbed case, however, this norm is not adequate. A different scaling of the H^m seminorm for $2m$-th order problems leads to a balanced norm which reflects the layer behaviour correctly. We prove error estimates in such balanced norms and improve thereby existing estimates known in literature.

Key words. Balanced norms, reaction-diffusion problems, finite element methods.

1. Introduction

We shall examine the finite element method for the numerical solution of a singularly perturbed linear elliptic $2m-$th order boundary value problem in two dimensions. In the weak form it is given by

$$\varepsilon^{2k}(\nabla^m u, \nabla^m v) + \tilde{a}(u,v) = (f,v) \quad \forall v \in H_0^m(\Omega), \tag{1}$$

where $\Omega = (0,1)^2$, $0 < \varepsilon \ll 1$ is a small positive parameter, $1 \leq k \leq m$ and f is sufficiently smooth. We assume that the bilinear form $\tilde{a}(\cdot,\cdot)$ is related to a $2(m-k)-$th order operator and $\tilde{a}(u,u)$ is equivalent to $\|u\|^2_{H^{m-k}}$.

The Lax-Milgram theorem tells us that the problem has a unique solution $u \in H_0^m(\Omega)$ which is sufficiently smooth for smooth data and satisfies in the energy norm

$$|||u|||_\varepsilon := \varepsilon^k |u|_{H^m} + \|u\|_{H^{m-k}} \lesssim \|f\|_{L^2}. \tag{2}$$

Here and in the following we use the following notation: if $A \lesssim B$ then there exists a (generic) constant C independent of ε (and later also of the mesh used) such that $A \leq C\,B$.

The error of a finite element approximation $u^N \in V^N$ satisfies

$$\left|\left|\left|u - u^N\right|\right|\right|_\varepsilon \lesssim \min_{v^N \in V^N} \left|\left|\left|u - v^N\right|\right|\right|_\varepsilon \tag{3}$$

for any finite dimensional space $V^N \subset H_0^m(\Omega)$.

If we use C^{m-1}-splines, piecewise polynomial of degree $2m-1$, on a properly defined Shishkin mesh with N cells in each direction, then one can prove for the interpolation error of the Hermite interpolant $u^I \in V^N$

$$\left|\left|\left|u - u^I\right|\right|\right|_\varepsilon \lesssim \left(\varepsilon^{1/2}(N^{-1}\ln N)^m + N^{-(m+1)}\right). \tag{4}$$

It follows that the error $u - u^N$ also satisfies such an estimate. Some special one-dimensional cases are discussed, for instance, in [4, 14, 15].

Received by the editors February 26, 2019 and in revised form April 17, 2020.
2000 *Mathematics Subject Classification.* 65N12, 65N15, 65N30.

However, a typical boundary layer function $\varepsilon^{m-k}\exp(-x/\varepsilon)$ of our given problem measured in the norm $|||\cdot|||_\varepsilon$ is of order $\mathcal{O}\left(\varepsilon^{1/2}\right)$. Consequently, error estimates in this norm are less valuable as for convection diffusion equations. Therefore, we ask the fundamental question:

Is it possible to prove error estimates in the balanced norm

$$|||v|||_b := \varepsilon^{k-1/2}|v|_{H^m} + \|v\|_{H^{m-k}} \quad ? \tag{5}$$

As this norm has a different weighting of the H^m-seminorm, the layer function is measured of order $\mathcal{O}(1)$ as well as the non-layer components of the solution – the norm is balanced.

For higher order equations ($m \geq 2$), even in 1d nothing is known concerning estimates in the balanced norm for the Galerkin finite element method. The only exception is [2], where a fourth-order problem is discretised with a mixed finite element method.

The outline of this paper is as follows. In Section 2 we present a new idea to derive balanced error estimates for second order problems, improving the result in [11]. In Section 3 we generalise the idea from Section 2 to higher order problems in detail for the 1d case and give guiding principles for the (very technical) 2d case.

Notation: We denote by $(\cdot,\cdot)_D$ the L^2-scalar product on D and by $\|\cdot\|_{L^2(D)}$ the associated L^2-norm over D. Furthermore by $|\cdot|_{H^k(D)}$, $\|\cdot\|_{H^k(D)}$ and $\|\cdot\|_{W^{k,\infty}(D)}$ we denote the Sobolev-seminorm and norms in $H^k(D) = W^{k,2}(D)$ and $W^{k,\infty}(D)$. In the case of $D = \Omega$ we may skip the reference to the domain.

2. An improved estimate in a balanced norm for second order problems

Let us consider the case $m = k = 1$ and the discretization of

$$\varepsilon^2(\nabla u, \nabla v) + (cu, v) = (f, v) \quad \forall v \in V = H_0^1(\Omega), \tag{6}$$

where $c \geq \gamma > 0$ by linear finite elements on S-type meshes [10]. In [11] it was proved (on a Shishkin mesh)

$$\left|\left|\left|u - u^N\right|\right|\right|_b \lesssim N^{-1}(\ln N)^{3/2} + N^{-2}. \tag{7}$$

It was an open question to remove the factor $(\ln N)^{1/2}$ from (6). Here we modify the technique from [11] to realise that goal and use the same technique in Section 3 for higher order problems.

In [11] the L^2-projection $\pi u \in V^N$ from u was used instead of the Lagrange interpolant. Based on

$$u - u^N = u - \pi u + \pi u - u^N$$

we estimated for constant c the discrete error $\pi u - u^N$ starting from:

$$\begin{aligned}\left|\left|\left|\pi u - u^N\right|\right|\right|_\varepsilon^2 &\lesssim \varepsilon^2\|\nabla(\pi u - u^N)\|_{L^2}^2 + c\,\|\pi u - u^N\|_{L^2}^2\\ &= \varepsilon^2(\nabla(\pi u - u), \nabla(\pi u - u^N)) + c\,(\pi u - u, \pi u - u^N).\end{aligned} \tag{8}$$

With $(\pi u - u, \xi) = 0$ for $\xi \in V^N$, the last term vanishes and the problem was to estimate $\|\nabla(\pi u - u)\|_{L^2}$. The use of the global projection leads to difficulties, especially in 2D: it is known that the L^2 projection is not on every mesh L^p stable, and there are examples which show that for the $W^{1,p}$ stability restrictions on the mesh are necessary even in the one-dimensional case [1,7]. Fortunately, on tensor product meshes like our S-type meshes (and their triangular versions) the L^2-projection is