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# SUPERLINEAR CONVERGENCE OF AN SQP-TYPE METHOD FOR NONLINEAR SEMIDEFINITE PROGRAMMING

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**Abstract.** In this paper, we study the rate of convergence of a sequential quadratic programming (SQP) method for nonlinear semidefinite programming (SDP) problems. Since the linear SDP constraints does not contribute to the Hessian of the Lagrangian, we propose a reduced SQP-type method, which solves an equivalent and reduced type of the nonlinear SDP problem near the optimal point. For the reduced SDP problem, the well-known and often mentioned "$\sigma$-term" in the second order sufficient condition vanishes. We analyze the rate of local convergence of the reduced SQP-type method and give a sufficient and necessary condition for its superlinear convergence. Furthermore, we give a sufficient and necessary condition for superlinear convergence of the SQP-type method under the nondegeneracy condition, the second-order sufficient condition with $\sigma$-term and the strict complementarity condition.

**Key words.** Nonlinear semidefinite programming, SQP-type method, second order sufficient condition, constraint nondegeneracy, superlinear convergence.

## 1. Introduction

Consider the following nonlinear semidefinite programming (SDP) problem

$$
\begin{array}{lll}
 & \min\limits_{x\in\mathcal{R}^n} & f(x) \\
(1) & \text{s.t.} & h(x) = 0, \\
 & & G(x) \succeq 0,
\end{array}
$$

where $f : \mathcal{R}^n \to \mathcal{R}$, $h : \mathcal{R}^n \to \mathcal{R}^l$ and $G : \mathcal{R}^n \to \mathcal{S}^m$ are all smooth functions. $\mathcal{S}^m$ denotes the linear space of $m$-order real symmetric matrices, $\mathcal{S}^m_+$ and $\mathcal{S}^m_{++}$ denote the linear space of $m$-order real symmetric positive semidefinite matrices and symmetric positive definite ones, respectively. $\succeq$ and $\succ$ denote positive semidefinite order and positive definite order, which means $A \succeq B$ if and only if $A - B \in \mathcal{S}^m_+$ and $A \succ B$ if and only if $A - B \in \mathcal{S}^m_{++}$, respectively. In the past few years, basic theoretical issues of nonlinear semidefinite programming have been studied, such as optimality conditions ([9, 15]), duality theory ([6]), stability analysis ([1, 7, 10]) and so forth.

There are various methods for solving nonlinear SDP problem, such as the penalty/barrier multiplier method ([13]), the Augmented Lagrangian method ([11, 12]), the primal-dual interior point method ([21]), sequential semidefinite programming (SSDP) method ([3, 22, 23]) and so forth. As one of effective methods for solving nonlinear SDP problem, the SSDP method is a development of the SQP-type method on semidefinite cone space. The main idea of such method is to generate steps by solving a sequence of quadratic semidefinite subproblems. At the current iterate $x_k$, the trial step $d_k$ is obtained by solving the following quadratic

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semidefinite programming subproblem

$$
\begin{aligned}
\min_{d\in\mathcal{R}^n} \quad & \nabla f(x_k)^T d + \frac{1}{2} d^T W_k d \\
\text{s.t.} \quad & h(x_k) + Dh(x_k)d = 0, \\
& G(x_k) + DG(x_k)d \succeq 0,
\end{aligned}
\tag{2}
$$

where $\nabla f(x)$ denotes the gradient of the objective function $f(x)$, $W_k$ denotes the Hessian matrix of the Lagrangian function of (1) (see (3) for its definition) or its approximate symmetric matrix, $Dh(x)$ denotes the Jacobian matrix of $h(x)$ and

$$Dh(x)^T = (\nabla h_1(x), \nabla h_2(x), \cdots, \nabla h_l(x)).$$

Linear operator $DG(x)$ denotes by

$$DG(x) = \left( \frac{\partial G(x)}{\partial x_1}, \frac{\partial G(x)}{\partial x_2}, \cdots, \frac{\partial G(x)}{\partial x_n} \right),$$

which satisfies

$$DG(x)d = \sum_{i=1}^{n} d_i \frac{\partial G(x)}{\partial x_i}, \quad \forall d \in \mathcal{R}^n.$$

Hence, the new iteration is updated by a line search $x_{k+1} = x_k + \alpha_k d_k$, where $\alpha_k \in (0, 1]$ is a step size.

Under suitable assumptions, the sequence generated by the algorithm above converges globally to a KKT point of the problem (1), e.g., see [3, 23]. A fast local rate of convergence can be arrived if $W_k$ is a good approximation of the Hessian matrix. Fares et al. ([8]) proved the local quadratic rate of convergence under the maximal rank condition and the second order sufficient condition without $\sigma$-term. Under the nondegeneracy condition and the second-order sufficient condition with $\sigma$-term, Wang et al. ([19]) made a further analysis on the local property of convergence for SSDP method and also proved that the algorithm has a local quadratic convergence rate when $W_k$ is chosen as the Hessian matrix of the Lagrangian function. Zhao and Chen ([23]) gave a globally convergent algorithm based on the references [3, 8, 19] and proved the step size $\alpha_k$ of the algorithm is always equal to 1 for $k$ sufficiently large under the nondegeneracy condition and the second-order sufficient condition with $\sigma$-term. Thus the algorithm is convergent superlinearly. The results on the local rate of convergence of some other methods, such as the primal-dual interior point method, the augmented Lagrangian method etc., for nonlinear SDP can be found in [14, 16, 18, 20].

It's worth noting that the results on the local convergence mentioned above all include multiplier term. There are few researches on the convergent rate of the sequence without multiplier term which is generated by SQP-type method for nonlinear SDP, while Boggs et al. ([2]) has already proposed an sufficient and necessary condition in the case of nonlinear programming in the 1980s. Another point of attention is that the Hessian matrix of the Lagrangian function at the optimal point is not necessarily positive definite on the critical cone (see the example showed in [4]). Therefore the subproblems in the SQP-type method near the optimal point may be nonconvex when $W_k$ takes the Hessian matrix of the Lagrangian function of (1) or its approximate symmetric matrix, which may influence the local convergence properties of the method. In this paper, we analyze the convergent rate of the sequence without multiplier term. And then an equivalent and reduced type of the primal problem near the optimal point is analyzed and the conditions of superlinear convergence are discussed. Finally, a sufficient and necessary condition for superlinear convergence of the algorithm is given under the nondegeneracy condition,