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FITTED FRONT TRACKING METHODS FOR TWO-PHASE INCOMPRESSIBLE NAVIER–STOKES FLOW: EULERIAN AND ALE FINITE ELEMENT DISCRETIZATIONS

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Abstract. We investigate novel fitted finite element approximations for two-phase Navier–Stokes flow. In particular, we consider both Eulerian and Arbitrary Lagrangian–Eulerian (ALE) finite element formulations. The moving interface is approximated with the help of parametric piecewise linear finite element functions. The bulk mesh is fitted to the interface approximation, so that standard bulk finite element spaces can be used throughout. The meshes describing the discrete interface in general do not deteriorate in time, which means that in numerical simulations a smoothing or a remeshing of the interface mesh is not necessary. We present several numerical experiments, including convergence experiments and benchmark computations, for the introduced numerical methods, which demonstrate the accuracy and robustness of the proposed algorithms. We also compare the accuracy and efficiency of the Eulerian and ALE formulations.

Key words. Incompressible two-phase flow, Navier–Stokes equations, ALE method, free boundary problem, surface tension, finite elements, and front tracking.

1. Introduction

Fluid flow problems with a moving interface are encountered in many applications in physics, engineering and biophysics. Typical applications include drops and bubbles, die swell, dam break, liquid storage tanks, ink-jet printing and fuel injection. For this reason, developing robust and efficient numerical methods for these flows is an important problem and has attracted tremendous interest over the last few decades.

A crucial aspect of these types of fluid flow problems is that apart from the solution of the flow in the bulk domain, the position of the interface separating the two bulk phases also needs to be determined. At the interface certain boundary conditions need to be fulfilled, which specify the motion of the phase boundary. These conditions relate the variables of the bulk flow, velocity and pressure, across the two phase, taking into account external influences, such as for example surface tension. Numerically, in order to be able to compute the flow solution as well as the interface geometry, a measure to track the interface starting from an initial position needs to be incorporated. There are several strategies to deal with this problem, which can be divided into two categories depending on the viewpoint: interface capturing and interface tracking.

Interface capturing methods use an Eulerian description of the interface, which is defined implicitly. A characteristic scalar field, that is advected by the flow, is used to identify the two phases as well as the interface along the boundaries of the individual fluid domains. Depending on the method, this scalar field may be, for example, a discontinuous Heaviside function or a signed-distance function. The most important methods, which belong to this category, are the volume-of-fluid method, the level-set method and the phase-field method. In the volume-of-fluid method, the characteristic function of one of the phases is approximated

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numerically, see e.g. [38, 50, 49]. In the level-set method, the interface is given as the level set of a function, which has to be determined, see e.g. [54, 53, 47, 34, 35, 55]. Instead, the phase-field method works with diffuse interfaces, and therefore the transition layer between the phases has a finite size. There is no tracking mechanism for the interface, but the phase state is included implicitly in the governing equations. The interface is associated with a smooth, but highly localized variation of the so-called phase-field variable. Examples for phase-field methods applied to two-phase flow are [39, 5, 44, 15, 27, 22, 42, 1, 4, 36, 32]. Extensions of the method to multi-phase flows can be found in [24, 25, 16, 6]. The appeal of interface capturing approaches is the fact that they are usually easy to implement, and that they offer an automated way to deal with topological changes.

Interface tracking approaches, instead, use a Lagrangian description of the interface which is described explicitly. Here the interface is represented by a collection of particles or points, and this representation is transported by the bulk flow velocity. The great advantage of interface tracking approaches is that they offer an accurate and computationally efficient approximation of the evolving interface. Challenges are the mesh quality both of the interface representation and of the bulk triangulations, as well as the need to heuristically deal with topological changes. We refer e.g. to [57, 7, 56, 31, 10, 12] for further details, and to [43, 48] for the related immersed boundary method, which is used to simulate fluid-structure interactions using Eulerian coordinates for the fluid and Lagrangian coordinates for the structure.

In this paper we use a direct description of the interface using a parametrization of the unknown interface, similarly to the previous work [12]. In particular, our numerical method will be based on a piecewise linear parametric finite elements, and the description of the interface will be advected in normal direction with the normal part of the fluid velocity. The tangential degrees of freedom of the interface velocity are implicitly used to ensure a good mesh quality, and this is a main feature of our proposed methods. But in contrast to [12], where an unfitted bulk finite element approximation was used, we will adopt a fitted mesh approach, which means that the discrete interface is composed of faces of elements from the bulk triangulation. Fitted and unfitted bulk mesh approaches are fundamentally different approximation methods, and need specialized implementation techniques. A fitted method has the advantage that discontinuity jumps at the interface are captured naturally, but it has the disadvantage that in a standard Eulerian method the velocity needs to be interpolated from an old mesh to a new mesh, unless the interface is stationary. Hence so-called Arbitrary Lagrangian Eulerian (ALE) methods are often proposed. Here the equations are posed in a moving domain framework, and an arbitrary Lagrangian velocity may be chosen to improve the quality of the bulk mesh. The original idea for ALE methods goes back to the papers [23, 40], and we refer to [46, 45, 28, 29, 31, 37, 30] for applications to two-phase Navier–Stokes flow.

In this paper we will propose both standard Eulerian and ALE finite element approximations for two-phase incompressible Navier–Stokes flow. One aim of our paper is to investigate numerically, if there is a clear advantage of ALE type methods over standard Eulerian methods. We stress that we are not aware of any detailed comparisons between fitted Eulerian and ALE front tracking methods in the literature. This paper aims to fill this gap. On the other hand, similar comparisons between ALE type interface tracking methods and various interface capturing methods have been presented in e.g. [41, 26]. We note that in the case of viscous incompressible two-phase Stokes flow, there is no need for the numerical method