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WEAK GALERKIN METHOD FOR THE HELMHOLTZ EQUATION WITH DTN BOUNDARY CONDITION

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Abstract. In this article, we consider a weak Galerkin finite element method for the two dimensional exterior Helmholtz problem. After introducing a nonlocal boundary condition by means of the exact Dirichlet to Neumann (DtN) operator for the exterior problem, we prove that the existence and uniqueness of the weak Galerkin finite element solution for this problem. Then, applying some projection techniques, we establish a priori error estimate, which include the effect of truncation of the DtN boundary condition as well as the spatial discretization. Finally, some numerical examples are presented to confirm the theoretical predictions.

Key words. Helmholtz equation, weak Galerkin method, Dirichlet to Neumann operator, error estimates.

1. Introduction

We consider the exterior Helmholtz problem:

$$-\Delta u - k^2 u = f \quad \text{in}\, \Omega^c, \tag{1}$$

$$u = g \quad \text{on}\, \Gamma_0, \tag{2}$$

$$\lim_{r\to+\infty} r^{1/2}\left(\frac{\partial u}{\partial r} - \mathrm{i}ku\right) = 0 \quad r = |x|, \tag{3}$$

where i denotes the imaginary unit and $k \in \mathbb{R}$ is known as the wave number. $\Omega \subset \mathbb{R}^2$, be an open, bounded domain with smooth boundary Γ_0, and let $\Omega^c = \mathbb{R}^2\backslash\overline{\Omega}$ be the unbounded exterior domain in $\mathbb{R}^2$. The equation (3) is known as the standard Sommerfeld radiation condition.

In order to solve the exterior problem in an infinite domain numerically, it is usual to limit the computation to a finite domain by introducing an artificial boundary. The original exterior domain problem reduced to a boundary value problem by enforcing a boundary condition on the artificial boundary. In [5, 7, 11, 12, 13, 14, 17, 19, 21, 29, 31, 33, 43, 46, 52], the authors developed different numerical methods for the Helmholtz equation with the lowest order absorbing boundary condition. However, the lowest order absorbing boundary conditions will lead to large errors caused by the reflections from the artificial boundary, unless the computational domain is large. To decrease the errors caused by boundary reflection, several numerical approaches have been considered. These contain high order absorbing boundary condition [16], perfectly matched layer [3], and the Dirichlet to Neumann (DtN) boundary condition [18, 22, 24, 25, 44] and the references therein. The DtN boundary condition is called exact nonreflecting boundary condition because it allows waves to propagate outward without producing any spurious reflection from the artificial boundary. The DtN condition was derived by Keller and Givoili [23], for the Helmholtz equation when the artificial boundary is a circle or sphere. In [24], Koyama established a priori error estimates for the DtN finite element method, including the effects of truncation of the DtN boundary condition and the finite

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element discretization. Hsiao et. al [18] derived for the DtN finite element method and present numerical results that show optimal convergence in the L^2 and H^1 norms using conforming piecewise linear finite elements. Kapita and Monk, in [22], considered an approximate boundary value problem with a truncated DtN series by the plane wave discontinuous Galerkin method and derived error estimates with respect to the truncation order of the DtN map and mesh width. In [44], Wang et. al coupled the DG method and spectral method to solve the DtN boundary value problem and gave DG-norm and L^2 norm errors analysis explicitly with wave number.

The numerical studies of the Helmholtz equation have been become an extraordinary popular research field in recent years. Many numerical schemes have been developed and analyzed for the Helmholtz equation, for instance, conforming finite element method [19, 21], interior penalty discontinuous Galerkin method [11], continuous interior penalty Galerkin method [46, 52], local discontinuous Galerkin method [12], hybridizable discontinuous Galerkin method [5, 14], plane wave discontinuous Galerkin method [13, 17, 22], weak Galerkin(WG) finite element method [7, 31, 33, 43], spectral method [29], finite difference method [15, 39, 40, 41, 42], immersed finite element method [26, 27] and the references therein. Weak Galerkin finite element methods were first introduced by Wang and Ye in [37] for second order elliptic equations. The main idea behind weak Galerkin method lies in the classical gradient operator replaced by a discrete weak gradient operator for weak functions on a partition of the domain. Weak Galerkin methods have been widely used to solve a variety of partial differential equations[34, 35, 36, 45, 47, 48, 49, 50, 51]. The methods have been applied for solving the Helmholtz equation with the lowest order absorbing boundary condition in [7, 31, 33, 43]. In [33], due to use of the RT and BDM elements, the WG finite element methods were limited to classical finite element partitions. Following the stabilization technique of [38], Mu et. al developed a new WG method that admits general finite element partitions with a mix of arbitrary shape of polygons and polyhedrons.

In practical computations, the use of the Fourier series representation of the DtN operator requires truncating the infinite series at a finite order to obtain an approximation DtN operator. So in this paper, we firstly investigate the truncating order of the series. Then, after studying the Gårding inequality and the unique of the weak solution, we establish a priori error estimates in the tri-norm and L^2 norm for the exterior Helmholtz problem by using weak Galerkin finite element method, including the effect of the truncation order and the finite element discretization. Following the stabilization approach of [38], we also add a parameter-free stabilier in the weak Galerkin formulation. The present weak Galerkin method is more flexible in terms of choosing approximation functions and finite element partitions. What's more, it is well suited to handling obstacles with complicated geometries and is easy to handle DtN boundary condition.

An outline of the remainder of the paper follows: In Section 2, we describe the nonlocal boundary value problem and modified nonlocal boundary value problem. Section 3 is devoted to a description of weak Galerkin method and algorithm. We shall design a weak Galerkin formulation for the modified nonlocal boundary value problem given in (10)–(12) and demonstrate the well-posedness of the formulation. We present an error equation and the error estimates in Section 4. In Section 5, we report some numerical results in order to demonstrate the accurate and efficient of our method. In this section, we consider the first approximation of the DtN operator. The reason is that, the first approximation will generate a sparse linear