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LOCALLY CONSERVATIVE FINITE ELEMENT SOLUTIONS FOR PARABOLIC EQUATIONS

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Abstract. In this paper, we post-process the finite element solutions for parabolic equations to meet discrete conservation laws in element-level. The post-processing procedure are implemented by two different approaches : one is by computing a globally continuous flux function and the other is by computing the so-called finite-volume-element-like solution. Both approaches only require to solve a small linear system on each element of the underlying mesh. The post-processed flux converges to the exact flux with optimal convergence rates. Numerical computations verify our theoretical findings.

Key words. Conservation laws, postprocessing, finite volume solution.

1. Introduction.

We consider numerical solutions of the following spatially-two-dimensional parabolic equations :

$$
(1) \qquad \begin{cases} u_t - \nabla \cdot (\kappa(\mathbf{x})\nabla u) = f(\mathbf{x},t), \quad (\mathbf{x},t) \in \Omega \times (0,T], \\ u = 0, \quad \mathbf{x} \in \partial\Omega \times (0,T], \\ u(\mathbf{x},0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \end{cases}
$$

where Ω is a convex bounded polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. We assume that $f(\mathbf{x},t) \in L^2(\Omega)$ for $t \in [0,T]$ and the coefficient function κ is Lipschitz continuous, there exits two positive constants κ_* and κ^* such that $\kappa_* \leqslant \kappa(\mathbf{x}) \leqslant \kappa^*$ for almost all $\mathbf{x} \in \Omega$. The above parabolic equations are widely used in the modelling of physical phenomena such as that from hydrological, biological and biogeochemical disciplines [6, 10, 19, 30]. Due to the lack of analytical solution and the expensive cost of physical experiments, numerical simulations received a great deal of attention in the study of parabolic problems. Among all numerical methods, those who guarantee locally conservation laws received a great deal of attention.

The finite volume method (FVM, see e.g.,[3, 11, 13, 17, 16])) is an important numerical method which preserves the conservation law in element level, it is very popular in computational fluid dynamics(CFD, see e.g., [24]). However, the linear algebraic system resulting from the FVM is generally non-symmetric, its implementation and analysis is challenging, especially for high order schemes(c.f., [2, 12, 18, 25, 26, 29]). The linear system derived from the finite element method(FEM) is symmetric and thus can be computed with many fast solvers. The FEM solutions, however, do not satisfy the local conservation laws. Therefore, many efforts have been made to post-process FEM solutions to derive solutions which satisfy local conservation laws during the past several decades. To the best of our knowledge, the first work on post-processing of the FEM solutions to derive locally conservative fluxes can be traced back to Douglas, Dupont and Wheeler([9]), which is designed in 1974 for elliptic equations. Thereafter a lot of works along this direction are reported in the literature. For instances, in 2006, Bochev and Gunzburger

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develop a flux-correction procedure for the Darcy flow equation based on the least-squares method, their derived solution guarantees local conservation law without compromising its L^2 accuracy ([5]). In 2007, Cockburn et al. present a two-step post-processing algorithm to generate a conservative flux([8]). In 2013, Pouliot et al. post-process the FEM solutions based on the flux superconvergent-points([21]). In [28] , Zhang et al. develop elemenwisely conservative flux by correcting the FEM solution element-by-element. In [32], Zou et al. derive volume-wisely conservative flux by solving a small linear system in each element of the underlying mesh.

In the present paper, we apply the post-processing techniques in [32] to parabolic equations. As that in [32], our post procedure here can be implemented element-wisely. Moreover, our post processing techniques share almost all advantages possessed by the techniques in [32]. For instances, the post-precessed numerical flux satisfies the local conservation law and converges to the exact flux with optimal orders, etc. However, since the parabolic equation is related to the time evolution, our post-processing procedure here is significantly different from that for elliptic equations by solving an ordinary differential equation system in each element of the underlying mesh instead of solving a static linear system in each element of the underlying mesh which is done in [32].

The rest of the paper is organized as below. In Section 2, we present semi-discrete FEM and FVM solutions and their related properties. In Section 3, we post-process the semi-discrete FEM solution to obtain a globally continuous flux function and a finite-volume-element-like solution, both locally conservative. The approximation property of the post-processed solution will be also discussed. In Section 4, we illustrate how to implement our post-processing techniques in practical algorithms associated with a certain temporal discretization. In Section 5, several numerical experiments are made to demonstrate the efficiency and accuracy of our post-processing algorithms.

We close the section by an introduction of some notation. Let $D \subset \mathbb{R}^2$ be an open bounded domain with Lipschitz continuous boundary. We adopt standard notations for Sobolev spaces such as $W^{m,p}(D)$ on sub-domain $D \subset \Omega$ equipped with the norm $\|\cdot\|_{m,p,D}$ and semi-norm $|\cdot|_{m,p,D}$. When $D = \Omega$, we omit the index D; and if $p = 2$, we set $W^{m,p}(D) = H^m(D)$, $\|\cdot\|_{m,p,D} = \|\cdot\|_{m,D}$, and $|\cdot|_{m,p,D} = |\cdot|_{m,D}$. Notation $A \lesssim B$ implies that A can be bounded by B multiplied by a constant independent of the mesh size h. $A \simeq B$ means that both $A \lesssim B$ and $B \lesssim A$.

2. Semi-discrete finite element and finite volume solutions.

To illustrate our basic idea on post-processing, we only present the semi-discrete schemes instead of fully-discrete schemes for (1) in this section. We begin our presentation with an introduction on the spatial discretization. Let $\mathcal{T}_h = \{\tau\}$ be a family of quasi-uniform and shape-regular *triangulation* on Ω. We denote by $\mathcal{N}_h$, $\mathring{\mathcal{N}}_h$, $\mathcal{E}_h$, $\mathring{\mathcal{E}}_h$ the set of all vertices, the set of internal vertices, the set of all edges, and the set of internal edges, respectively. Let the standard linear finite element space be

$$V_h = \{v \in C(\overline{\Omega}) : v|_\tau \in \mathcal{P}_1, \forall \tau \in \mathcal{T}_h, v|_{\partial\Omega} = 0\},$$

where $\mathcal{P}_1$ is the space of all first-order polynomials. It's known that $V_h \subset H_0^1(\Omega)$ and it has a standard Lagrange basis $S_h(T) = span\{\phi_P, P \in \mathring{\mathcal{N}}_h\}$, where $\phi_P \in V_h$ is nodal basis function satisfying $\phi_P(P') = \delta_{PP'}$.